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Approximate method for redundancy allocation problem in multi-state series-parallel system

Keywords

reliability optimization, redundancy allocation problem, dynamic programming

Abstract

Redundancy allocation problem (RAP) is one of the most important model of reliability optimization problems. In literature, there exist two typical formulations for RAP, namely 1) maximization of the system reliability under the resource constraints and 2) minimization of the system cost under system reliability and resource constraints. We propose an approximate method to solve MSSPS RAP under homogeneous and heterogeneous conditions under an absolute error. Numerical experiments are conducted on five published benchmarks and a new randomly generated instance with a larger number of subsystems. The approximated solutions exhibit good quality throughout the experiments.

1. Introduction

Reliable systems are important to industrial practitioners and engineering. There are generally four methods to improve the system reliability (Coit & Zio, 2019):

(i) increasing the component reliability,
(ii) paralleling the redundant components,
(iii) combination of (i) and (ii),
(iv) reassignment of interchangeable components.

Redundancy allocation problem (RAP) is an important problem in reliability optimization, which determines both the redundancy levels and select the version (type) of components (Mellal et al., 2020; Wang et al., 2020).

The traditional system structure considered in reliability optimization is the binary-state system (BSS), where the systems have two possible states: perfect working and total failure. However, binary-state assumption cannot completely capture the characteristic of the sophisticated degradation process of engineering component or system. By introducing the intermediate states between perfect working state and total failure state, multi-state models reflect the stochastic performance of a system/component in a more accurate way. The system with multi-state degradation is generally considered as a multi-state system (MSS). There are many applications for MSS in different fields, such as gas pipeline system (Bao et al., 2021), wireless sensor system (Yi et al., 2021), distributed generation system (Li & Zio, 2012), wind energy system (Eryilmaz, 2018) and mineral water production system (Hao et al., 2020). The common system structure considered in MSS is series-parallel system, i.e. multi-state series-parallel system (MSSPS).

The literature about MSSPS RAP in the past 5 years can be classified into the following groups. The first group considered the different redundancy strategies in RAP, such as mixed-strategy of active and cold-standby redundancies (Ardakan et al., 2016), and warm-standby (Levitin et al., 2017). The second group considered optimizing the component reliability joint with RAP

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(Ardakan et al., 2016; Muhuri & Nath, 2019; Ouyang et al., 2019; Zhang et al., 2022). The third group considered the maintenance action of component in RAP (Bei et al., 2017; Bei et al., 2019). The forth group of works considered the epistemic uncertainty of parameters in RAP and used the robust optimization method to deal with it (Zhang & Li, 2022).

As for the solving method for MSSPS RAP, several heuristic techniques have been proposed, such as GA (Levitin et al., 1998; Levitin et al., 1997), TS (Ouzineb et al., 2008, 2009) and QEA (Li & Zio, 2014). Li and Zhang (2022) proposed the exact algorithm, RMODP, for MSSPS RAP. The size of the state space influences both the computational time and memory requirements for RMODP algorithm. Therefore, we proposed an approximate algorithm for MSSPS RAP which can improve the computational rate and guarantee the accuracy of the result.

The chapter is structured as follows. Section 2 gives the description of MSSPS RAP and gives the mathematical formulations. Section 3 states the approximate algorithm for MSSPS RAP. In Section 4, experiments are conducted to illustrate the performance of the algorithm. Finally, the conclusion is presented in Section 5.

2. Problem description

2.1. Assumption

We consider MSSPS RAP under the homogeneous and heterogenous cases. In each subsystem, homogeneous means that components mixing is not allowed, and heterogeneous means that the component type can be mixed. Therefore, we have to determine both the number of component and the selection of component type(s). Following are the assumptions we considered in this chapter:

- the components and systems have multiple and finite states,
- the degradation processes of all component are independent,
- the characteristics of the component (reliability, cost) are known.

2.2. Mathematical formulation

MSSPS consists of N subsystems connected in series. Each subsystem is indexed with i and has n_i types of multi-state components available. The type of component in subsystem is index with j. The component with type j in subsystem i is denoted by U_{ij}, which is described by their performance $\boldsymbol{g}_{ij}$, unit cost c_{ij}, and reliability $\boldsymbol{p}_{ij}$. We assume that the number of component state for multi-state element U_{ij} is M_{ij} and it is index by m_{ij}. U_{ij} has the performance level $g_{ij}^{m_{ij}}$ with probability $p_{ij}^{m_{ij}}$ for $m_{ij} = 1, \dots, M_{ij}$ and $\sum_{m_{ij}} p_{ij}^{m_{ij}} = 1$. The decision variables are to determine the number of all components in the system. The number of U_{ij} is denoted by x_{ij} for $i = 1, \dots, N, j = 1, \dots, n_i$. Thus, the decision variable can be simplified to $\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$, where $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \dots, x_{in_i})$.

The system demand W is a random discrete variable and equals to W_l with the probability p_l' for $l = 1, \dots, L$. Let G_i denote the performance of subsystem i. Then, the reliability of subsystem i is the probability that the subsystem performance meets the system demand and given by $\Pr(G_i \geq W)$. The reliability of MSSPS is calculated by the probability that the system performance meets the system demand. Since the subsystems are connected in series, the reliability is the multiplication of all subsystem reliabilities and given by

$$R(\boldsymbol{x}) = \prod_{i=1}^{N} \Pr(G_i \geq W)$$

$$= \sum_{l=1}^{L} p_l' \prod_{i=1}^{N} \Pr(G_i \geq W_l). \tag{1}$$

The methods to evaluate the reliability of MSSPS include: the universal generating function (UGF) (Ushakov, 1988), the recursive algorithm (Li & Zuo, 2008), and the method based on the dynamic programming concept with a pseudo-polynomial time complexity (Sun et al., 2015). In this chapter, we use the dynamic programming method to calculate the system reliability.

According to the form of the objective function, MSSSP RAP can be divided into two types (Kuo & Wan, 2007): P1) maximization of the system reliability under the resource constraints and P2) minimization of the system cost under system reliability and resource constraints. The mathematical formulations of P1 and P2 are given by (2) and (3).

$$\text{P1} \quad \max \quad \sum_{l=1}^{L} p_l' \prod_{i=1}^{N} \Pr(G_i \geq W_l) \tag{2.a}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C_0 \tag{2.b}$$

$$\boldsymbol{x_i} \in X_i, i = 1, \dots, N. \tag{2.c}$$

$$\text{P2} \quad \min \quad \sum_{i=1}^{N} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C_0 \tag{3.a}$$

$$\text{s.t.} \quad \sum_{l=1}^{L} p_l' \prod_{i=1}^{N} \Pr(G_i \geq W_l) \geq R_0 \tag{3.b}$$

$$\boldsymbol{x_i} \in X_i, i = 1, \dots, N. \tag{3.c}$$

3. Approximate algorithm

We propose an approximate method based on the exact method, RMODP, proposed in (Li & Zhang 2022). The size of the state space influences both the computational time and memory requirements for RMODP algorithm. To handle this difficulty, we develop an approximate algorithm with a provable absolute error to solve P2, which can largely reduce the state space and save the computational time.

First, we convert the single objective function (2.a) into multi-objective functions (4.a), each objective function $R_l(\boldsymbol{x})$ is corresponding to one demand W_l. In this formulation, all objectives are separable with decision variables $\boldsymbol{x}_i$. Thus, P1 is transformed into P3 and then all objectives are logarithmized to the linear forms, i.e. in P4. The objective $\sum_{i=1}^{N} \ln \Pr(G_i \geq W_l)$ is simply represented by $\sum_{i=1}^{N} f_l(\boldsymbol{x}_i)$ for $l = 1, \dots, L$. The optimal solution of P1 must be included in the set of Pareto optimal solutions of P4. Thus, we solve P4 to obtain the Pareto optimal solutions and select the one which maximizes P1.

$$\begin{aligned} \text{P3} \quad \max \quad & R_1(\boldsymbol{x}) = \prod_{i=1}^{N} \Pr(G_i \geq W_1) \\ & \vdots \\ \max \quad & R_L(\boldsymbol{x}) = \prod_{i=1}^{N} \Pr(G_i \geq W_L) \end{aligned} \tag{4.a}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C_0 \tag{4.b}$$

$$\boldsymbol{x_i} \in X_i, i = 1, \dots, N. \tag{4.c}$$

$$\begin{aligned} \text{P4} \quad \max \quad & \sum_{i=1}^{N} \ln \Pr(G_i \geq W_1) \triangleq \sum_{i=1}^{N} f_1(\boldsymbol{x}_i) \\ & \vdots \\ \max \quad & \sum_{i=1}^{N} \ln \Pr(G_i \geq W_L) \triangleq \sum_{i=1}^{N} f_L(\boldsymbol{x}_i) \end{aligned} \tag{5.a}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \sum_{j=1}^{n_i} c_{ij} x_{ij} \leq C_0 \tag{5.b}$$

$$\boldsymbol{x_i} \in X_i, i = 1, \dots, N. \tag{5.c}$$

For a given accuracy $\varepsilon > 0$, the capacity state is partitioned into several intervals. Then, we adapt the RMODP to the partitioned capacity space. The DP-procedure consists of N+1 stages, i.e. $k = 0, 1, \dots, N$. Each stage k represents the first k subsystems has been determined. Capacity b denotes the total cost of the feasible solutions at each stage, i.e. $b = 0, 1, \dots, C_0$. Let all feasible solutions denoted with a set of states Q by,

$$Q := \{q(b,k): b = 0, 1, \dots, C_0, k = 0, 1, \dots, N\}$$

where the state $q(b,k)$ represents all feasible solutions satisfying $\sum_{i=1}^{k} \boldsymbol{c}_i \boldsymbol{x}_i = b$, i.e.

$$q(b,k) := \{\boldsymbol{x} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_N): \sum_{i=1}^{k} \boldsymbol{c}_i \boldsymbol{x}_i = b, \boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_N = 0, \boldsymbol{x}_i \in X_i \ \}.$$

Let $G(q(b,k))$ denote the set of all nondominated solutions corresponding to the subsystem reliabilities $\left(\sum_{i=1}^{k} f_1(\boldsymbol{x}_i), \ldots, \sum_{i=1}^{k} f_L(\boldsymbol{x}_i)\right)^T$ at the state $q(b,k)$. The recursive function of approximate algorithm is given by:

$$\begin{cases} G\big(q(b,0)\big) = \{\boldsymbol{0}\} \\ G\big(q(b,k)\big) = v\max\{G\big(q(b-\boldsymbol{c}_k\boldsymbol{x}_k, k-1)\big) \end{cases}$$

$$G\big(q(b-\boldsymbol{c}_k\boldsymbol{x}_k, k-1)\big) + \begin{pmatrix} f_1(\boldsymbol{x}_k) \\ \vdots \\ f_L(\boldsymbol{x}_k) \end{pmatrix}:$$

$$\boldsymbol{x}_k \in X_k^{HOMO}, b-\boldsymbol{c}_k\boldsymbol{x}_k \geq 0\}$$

$$b = 0,1,\ldots,C_0, k = 1,\ldots,N.$$

4. Experiments and results

Most literatures are focused on P2 and we compare our proposed method with the exact algorithm proposed in (Li & Zhang 2022) on six instances by solving P2 to illustrate its performance. The first five instances are classical benchmarks used for RAP (Levitin et al., 1998; Levitin et al., 1997; Lisnianski et al., 1996; Ouzineb et al., 2008). The last one is a newly generated problem with more subsystems connected in series. The upper bound $\tilde{c}$ is set by the local optimal solution solved by any heuristic method.

4.1. Benchmark

Instance 1–5. The first five instance are lev4-(4/6)-3 (Levitin et al., 1998), lev5-(4/9)-4 (Levitin et al., 1997), lis4-(7/11)-4 (Lisnianski et al., 1996), mas5-(4/5)-4 (Massim et al., 2005) and ouz6-(4/11)-4 (Ouzineb et al., 2008). For the notation xxxa-(b/c)-d, xxx, a, (b/c) and d denote the first three characters of the first author's name, the number of subsystems, the range of component types, the number of the system demand levels, respectively.

Instance 6. The last one is a new randomly generated instance with more subsystems and fewer component types denoted by zhang10-(3/3)-3.

4.2. Approximated D-RMODP results

We implement the proposed approximate algorithm to six instances. The absolute error ε varies among 0.1, 0.3 and 0.5. The approximated solutions for homogeneous and heterogeneous cases are presented in detail in Appendix, where Table 5, 6 and 7 give the homogeneous solutions; Table 8, 9 and 10 give the heterogeneous solutions for $\varepsilon = 0.1$, $\varepsilon = 0.3$ and $\varepsilon = 0.5$ respectively. The approximate results for homogeneous and heterogeneous cases are illustrated in Table 1 and 3. The actual errors between the approximate results and the exact solutions for homogeneous and heterogeneous cases are illustrated in Table 2 and 4.

We observe that the actual error generally increases with the given ε increasing. When $\varepsilon = 0.1$ or 0.3, the approximate algorithm often obtains the exact solutions, since the partition scale A is not big enough to eliminate the exact solutions. When $\varepsilon = 0.5$, the actual absolute errors are generally relative smaller compared with the given ε.

Table 1. Approximate results for homogeneous RAP

Instance	R_0	ε		
		0.1	0.3	0.5
lis4-(7/11)-4	0.910	14.88615	14.88615	14.88615
	0.920	15.07515	15.07515	15.07515
	0.940	17.80475	17.80475	17.86050
	0.950	20.04875	20.04875	20.10100
	0.960	21.15515	21.15515	21.29450
	0.970	21.90675	21.90675	21.90675
	0.980	22.65625	22.65625	23.00000
	0.990	24.30475	24.30475	24.40700
	0.999	26.95170	26.97310	26.95170
lev5-(4/9)-4	0.975	16.450	16.450	16.521
	0.980	16.520	16.562	16.633
	0.990	17.050	17.050	17.166
lev4-(4/6)-3	0.900	5.986	5.986	5.986
	0.960	7.303	7.303	7.303
	0.990	8.328	8.328	8.328
ouz6-(4/11)-4	0.975	11.197	11.241	11.241
	0.980	11.369	11.369	11.419
	0.990	12.764	12.764	12.764
mas5-(4/5)-4	0.975	52.818	52.818	52.818
	0.985	56.112	56.112	56.112
	0.990	59.982	59.982	59.982
li10-(3/3)-3	0.985	3.95	3.95	3.95
	0.990	4.09	4.11	4.11
	0.993	4.18	4.18	4.18
	0.998	4.46	4.46	4.46

Table 2. Actual errors of approximate results for homogeneous RAP

Instance	R_0	ε		
		0.1	0.3	0.5
lis4-(7/11)-4	0.910	0	0	0
	0.920	0	0	0
	0.940	0	0	0.05575
	0.950	0	0	0.05225
	0.960	0	0	0.13935
	0.970	0	0	0
	0.980	0	0	0.10225
	0.990	0	0	0.10225
	0.999	0	0.0214	0
lev5-(4/9)-4	0.975	0	0	0.071
	0.980	0	0.1	0.280
	0.990	0	0	0.116
lev4-(4/6)-3	0.900	0	0	0
	0.960	0	0	0
	0.990	0	0	0
ouz6-(4/11)-4	0.975	0	0.044	0.044
	0.980	0	0	0.050
	0.990	0	0	0
mas5-(4/5)-4	0.975	0	0	0
	0.985	0	0	0
	0.990	0	0	0
li10-(3/3)-3	0.985	0	0	0
	0.990	0	0.02	0.02
	0.993	0	0	0
	0.998	0	0	0

Table 3. Approximate results for heterogeneous RAP

Instance	R_0	ε		
		0.1	0.3	0.5
lis4-(7/11)-4	0.910	14.88615	14.88615	14.88615
	0.920	15.07515	15.07515	15.07515
	0.940	17.41850	17.41850	17.41850
	0.950	19.86105	19.86105	19.86105
	0.960	20.57040	20.57040	20.57040
	0.970	21.28855	21.28855	21.28855
	0.980	22.56240	22.56240	22.56240
	0.990	23.77875	23.77875	23.83100
	0.999	26.9194	26.9194	26.95170
lev5-(4/9)-4	0.975	12.855	12.855	12.912
	0.980	14.755	14.791	14.755
	0.990	15.870	15.870	15.885
lev4-(4/6)-3	0.900	5.423	5.423	5.423
	0.960	7.009	7.009	7.060
	0.990	8.180	8.180	8.180
ouz6-(4/11)-4	0.975	11.197	11.197	11.241
	0.980	11.369	11.369	11.369
	0.990	12.764	12.764	12.764
mas5-(4/5)-4	0.975	52.392	52.392	52.392
	0.985	55.348	55.395	55.348
	0.990	57.032	57.032	57.070
li10-(3/3)-3	0.985	-	3.92	3.93
	0.990	-	4.08	4.09
	0.993	-	4.16	4.16
	0.998	-	4.45	4.46

Table 4. Actual errors of approximate results for heterogeneous RAP

Instance	R_0	ε		
		0.1	0.3	0.5
lis4-(7/11)-4	0.910	0	0	0
	0.920	0	0	0
	0.940	0	0	0
	0.950	0	0	0
	0.960	0	0	0
	0.970	0	0	0
	0.980	0	0	0
	0.990	0	0	0.05225
	0.999	0	0	0.03230
lev5-(4/9)-4	0.975	0	0	0.057
	0.980	0	0.036	0
	0.990	0	0	0.015
lev4-(4/6)-3	0.900	0	0	0
	0.960	0	0	0.051
	0.990	0	0	0
ouz6-(4/11)-4	0.975	0	0	0.044
	0.980	0	0	0
	0.990	0	0	0
mas5-(4/5)-4	0.975	0	0	0
	0.985	0	0.047	0
	0.990	0	0	0.038
li10-(3/3)-3	0.985	-	0	0.010
	0.990	-	0	0.010
	0.993	-	0	0
	0.998	-	0	0.010

5. Conclusion

We consider two types of problems for MSSPS RAP: maximization of reliability under the system cost restriction and minimization of cost under the system reliability restriction.
We proposed the approximate approach to solve MSSPS RAP under the homogeneous and heterogeneous cases. Compared to the exact method proposed in (Li & Zhang 2022), the approximate algorithm saves both the computational time and memory requirements under the give absolute accuracy error.
The numerical experiments demonstrate the performance of our proposed algorithm.

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Appendix

Table 5. Optimal solutions of approximated D-MODP for homogeneous RAP with $\varepsilon = 0.1$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.942	17.80475
	0.950	0.952	20.04875
lis4-(7/11)-4	0.960	0.960	21.15515
	0.970	0.971	21.90675
	0.980	0.982	22.65625
	0.990	0.991	24.30475
	0.999	0.999	26.95170
	0.975	0.977	16.450
lev5-(4/9)-4	0.980	0.981	16.520
	0.990	0.994	17.050
	0.900	0.910	5.986
lev4-(4/6)-3	0.960	0.961	7.303
	0.990	0.992	8.328
	0.975	0.975	11.197
ouz6-(4/11)-4	0.980	0.980	11.369
	0.990	0.990	12.764
	0.975	0.977	52.818
mas5-(4/5)-4	0.985	0.985	56.112
	0.990	0.990	59.982

Table 5. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-
1(5)	7(1)	5(3)	2(5)	-	-
1(5)	3(2)	2(5)	2(5)	-	-
10(1)	5(3)	2(4)	3(5)	-	-
10(1)	3(3)	5(3)	3(5)	-	-
10(1)	3(3)	2(3)	2(5)	-	-
1(5)	3(3)	5(3)	2(5)	-	-
1(6)	3(4)	2(6)	3(6)	-	-
2(2)	3(2)	2(3)	7(3)	2(1)	-
2(2)	5(6)	2(3)	7(3)	2(1)	-
2(2)	3(2)	2(3)	7(3)	4(3)	-
4(1)	3(2)	1(3)	5(2)	-	-
2(2)	3(3)	1(3)	5(2)	-	-
1(3)	3(3)	1(3)	2(5)	-	-
3(5)	1(5)	2(4)	2(9)	3(2)	4(1)
3(4)	1(5)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
3(1)	7(5)	5(4)	4(3)	5(3)	-
3(1)	8(5)	4(2)	4(2)	6(3)	-
5(3)	7(5)	5(4)	4(3)	5(3)	-

Table 6. Optimal solutions of approximated D-MODP for homogeneous RAP with $\varepsilon = 0.3$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.942	17.80475
	0.950	0.952	20.04875
lis4-(7/11)-4	0.960	0.960	21.15515
	0.970	0.971	21.90675
	0.980	0.982	22.65625
	0.990	0.991	24.30475
	0.999	0.999	26.97310
	0.975	0.977	16.450
lev5-(4/9)-4	0.980	0.981	16.562
	0.990	0.994	17.050
	0.900	0.910	5.986
lev4-(4/6)-3	0.960	0.961	7.303
	0.990	0.992	8.328
	0.975	0.979	11.241
ouz6-(4/11)-4	0.980	0.980	11.369
	0.990	0.990	12.764
	0.975	0.977	52.818
mas5-(4/5)-4	0.985	0.985	56.112
	0.990	0.990	59.982
	0.985	0.987	3.95
li10-(3/3)-3	0.990	0.991	4.11
	0.993	0.993	4.18
	0.998	0.998	4.46

Table 6. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-
1(5)	7(1)	5(3)	2(5)	-	-
1(5)	3(2)	2(5)	2(5)	-	-
10(1)	5(3)	2(4)	3(5)	-	-
10(1)	3(3)	5(3)	3(5)	-	-
10(1)	3(3)	2(3)	2(5)	-	-
1(5)	3(3)	5(3)	2(5)	-	-
1(6)	3(4)	2(5)	3(7)	-	-
2(2)	3(2)	2(3)	7(3)	2(1)	-
2(2)	5(7)	2(3)	7(3)	2(1)	-
2(2)	3(2)	2(3)	7(3)	4(3)	-
4(1)	3(2)	1(3)	5(2)	-	-
2(2)	3(3)	1(3)	5(2)	-	-
1(3)	3(3)	1(3)	2(5)	-	-
3(4)	1(4)	2(5)	2(7)	3(2)	4(1)
3(4)	1(5)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
3(1)	7(5)	5(4)	4(3)	5(3)	
3(1)	8(5)	4(2)	4(2)	6(3)	
5(3)	7(5)	5(4)	4(3)	5(3)	

Table 7. Optimal solutions of approximated D-MODP for homogeneous RAP with $\varepsilon = 0.5$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
lis4-(7/11)-4	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.940	17.86050
	0.950	0.952	20.10100
	0.960	0.962	21.29450
	0.970	0.971	21.90675
	0.980	0.982	23.00000
	0.990	0.992	24.40700
	0.999	0.999	26.95170
lev5-(4/9)-4	0.975	0.978	16.521
	0.980	0.981	16.633
	0.990	0.995	17.166
lev4-(4/6)-3	0.900	0.910	5.986
	0.960	0.961	7.303
	0.990	0.992	8.328
ouz6-(4/11)-4	0.975	0.979	11.241
	0.980	0.980	11.419
	0.990	0.990	12.764
mas5-(4/5)-4	0.975	0.977	52.818
	0.985	0.985	56.112
	0.990	0.990	59.982
li10-(3/3)-3	0.985	0.987	3.95
	0.990	0.991	4.11
	0.993	0.993	4.18
	0.998	0.998	4.46

Table 7. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-
1(6)	7(1)	2(4)	3(6)	-	-
1(5)	3(2)	2(5)	3(6)	-	-
10(1)	2(4)	5(3)	3(6)	-	-
10(1)	3(3)	5(3)	3(5)	-	-
10(1)	7(2)	2(5)	3(6)	-	-
1(5)	7(2)	5(3)	3(6)	-	-
1(6)	3(4)	2(6)	3(6)	-	-
2(2)	3(2)	2(3)	7(4)	2(1)	-
2(2)	5(7)	2(3)	7(4)	2(1)	-
2(2)	3(2)	2(3)	7(4)	3(3)	-
4(1)	3(2)	1(3)	5(2)	-	-
2(2)	3(3)	1(3)	5(2)	-	-
1(3)	3(3)	1(3)	2(5)	-	-
3(4)	1(4)	2(5)	2(7)	3(2)	4(1)
3(4)	1(6)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
3(1)	7(5)	5(4)	4(3)	5(3)	
3(1)	8(5)	4(2)	4(2)	6(3)	
5(3)	7(5)	5(4)	4(3)	5(3)	

Table 8. Optimal solutions of approximated D-MODP for heterogeneous RAP with $\varepsilon = 0.1$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
lis4-(7/11)-4	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.941	17.41850
	0.950	0.950	19.86105
	0.960	0.961	20.57040
	0.970	0.970	21.28855
	0.980	0.981	22.56240
	0.990	0.990	23.77875
	0.999	0.999	26.9194
lev5-(4/9)-4	0.975	0.976	12.855
	0.980	0.980	14.755
	0.990	0.992	15.870
lev4-(4/6)-3	0.900	0.900	5.423
	0.960	0.963	7.009
	0.990	0.991	8.180
ouz6-(4/11)-4	0.975	0.975	11.197
	0.980	0.980	11.369
	0.990	0.990	12.764
mas5-(4/5)-4	0.975	0.978	52.392
	0.985	0.985	55.348
	0.990	0.990	57.032

Table 8. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-
1(5)	7(1)	2(5)	2(1) 3(4)	-	-
1(5)	3(2)	2(5)	2(3) 3(2)	-	-
10(1)	2(1) 5(2)	2(4)	2(1) 3(4)	-	-
10(1)	2(1) 5(2)	2(5)	2(3) 3(2)	-	-
10(1)	3(3)	2(5)	2(4) 3(1)	-	-
1(5)	1(1) 5(2)	2(5)	2(5)	-	-
1(6)	3(4)	2(5)	2(3) 3(3)	-	-
4(2) 6(1)	5(6)	1(1) 4(1)	7(3)	4(3)	-
4(2) 6(1)	3(2)	2(1) 3(2)	7(3)	3(1) 4(2)	-
4(2) 6(1)	3(2)	2(2) 3(1)	7(3)	4(3)	-
4(1)	3(2)	1(3)	3(1) 5(1)	-	-
1(3)	2(1) 3(2)	1(3)	3(1) 5(1)	-	-
1(3)	3(3)	1(3)	3(1) 4(2)	-	-

continuation

3(5)	1(5)	2(4)	2(9)	3(2)	4(1)
3(4)	1(5)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
1(3)	3(1) 5(5)	4(5)	3(3) 4(1)	3(3) 4(2)	
1(1) 3(3)	1(2) 3(1) 5(1)	4(5)	3(3) 4(1)	3(3) 4(2)	
1(1) 3(3)	3(4)	4(5)	2(2) 3(1) 4(1)	3(4) 4(2)	

Table 9. Optimal solutions of approximated D-MODP for heterogeneous RAP with $\varepsilon = 0.3$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
lis4-(7/11)-4	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.941	17.41850
	0.950	0.950	19.86105
	0.960	0.961	20.57040
	0.970	0.970	21.28855
	0.980	0.981	22.56240
	0.990	0.990	23.77875
	0.999	0.999	26.9194
lev5-(4/9)-4	0.975	0.976	12.855
	0.980	0.980	14.791
	0.990	0.992	15.870
lev4-(4/6)-3	0.900	0.900	5.423
	0.960	0.963	7.009
	0.990	0.991	8.180
ouz6-(4/11)-4	0.975	0.975	11.197
	0.980	0.980	11.369
	0.990	0.990	12.764
mas5-(4/5)-4	0.975	0.978	52.392
	0.985	0.985	55.395
	0.990	0.990	57.032

Table 9. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-
1(5)	7(1)	2(5)	2(1) 3(4)	-	-
1(5)	3(2)	2(5)	2(3) 3(2)	-	-
10(1)	2(1) 5(2)	2(4)	2(1) 3(4)	-	-
10(1)	2(1) 5(2)	2(5)	2(3) 3(2)	-	-
10(1)	3(3)	2(5)	2(4) 3(1)	-	-
1(5)	1(1) 5(2)	2(5)	2(5)	-	-

continuation

1(6)	3(4)	2(5)	2(3) 3(3)	-	-
4(2) 6(1)	5(6)	1(1) 4(1)	7(3)	4(3)	-
4(2) 6(1)	3(2)	2(1) 3(2)	9(6)	4(3)	-
4(2) 6(1)	3(2)	2(2) 3(1)	7(3)	4(3)	-
4(1)	3(2)	1(3)	3(1) 5(1)	-	-
1(3)	2(1) 3(2)	1(3)	3(1) 5(1)	-	-
1(3)	3(3)	1(3)	3(1) 4(2)	-	-
3(5)	1(5)	2(4)	2(9)	3(2)	4(1)
3(4)	1(5)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
1(3)	3(1) 5(5)	4(5)	3(3) 4(1)	3(3) 4(2)	
1(3)	3(1) 5(6)	2(3) 3(1)	2(3) 4(1)	3(4) 4(2)	
1(1) 3(3)	3(4)	4(5)	2(2) 3(1) 4(1)	3(4) 4(2)	

Table 10. Optimal solutions of approximated D-MODP for heterogeneous RAP with $\varepsilon = 0.5$

Problem name	R_0	$R(\boldsymbol{x})$	$C(\boldsymbol{x})$
lis4-(7/11)-4	0.910	0.914	14.88615
	0.920	0.920	15.07515
	0.940	0.941	17.41850
	0.950	0.950	19.86105
	0.960	0.961	20.57040
	0.970	0.970	21.28855
	0.980	0.981	22.56240
	0.990	0.990	23.83100
	0.999	0.999	26.95170
lev5-(4/9)-4	0.975	0.976	12.912
	0.980	0.980	14.755
	0.990	0.992	15.885
lev4-(4/6)-3	0.900	0.900	5.423
	0.960	0.964	7.060
	0.990	0.991	8.180
ouz6-(4/11)-4	0.975	0.979	11.241
	0.980	0.980	11.369
	0.990	0.990	12.764
mas5-(4/5)-4	0.975	0.978	52.392
	0.985	0.985	55.348
	0.990	0.990	57.070

Table 10. Continue

Best solution					
1	2	3	4	5	6
11(1)	7(1)	2(4)	3(5)	-	-
10(1)	7(1)	2(4)	3(5)	-	-

continuation

1(5)	7(1)	2(5)	2(1) 3(4)	-	-
1(5)	3(2)	2(5)	2(3) 3(2)	-	-
10(1)	2(1) 5(2)	2(4)	2(1) 3(4)	-	-
10(1)	2(1) 5(2)	2(5)	2(3) 3(2)	-	-
10(1)	3(3)	2(5)	2(4) 3(1)	-	-
1(5)	1(1) 5(2)	2(5)	3(6)	-	-
1(6)	3(4)	2(6)	3(6)	-	-
4(2) 6(1)	5(7)	1(1) 4(1)	7(3)	3(1) 4(2)	-
4(2) 6(1)	3(2)	2(1) 3(2)	7(3)	3(1) 4(2)	-
4(2) 6(1)	3(2)	2(2) 3(1)	7(3)	3(1) 4(2)	-
4(1)	3(2)	1(3)	3(1) 5(1)	-	-
1(3)	3(3)	1(3)	3(1) 5(1)	-	-
1(3)	3(3)	1(3)	3(1) 4(2)	-	-
3(4)	1(4)	2(5)	2(7)	3(2)	4(1)
3(4)	1(5)	2(5)	2(8)	3(2)	4(1)
3(4)	1(4)	2(4)	2(8)	3(2)	4(2)
1(3)	3(1) 5(5)	4(5)	3(3) 4(1)	3(3) 4(2)	
1(1) 3(3)	1(2) 3(1) 5(1)	4(5)	3(3) 4(1)	3(3) 4(2)	
1(1) 3(3)	3(4)	4(5)	2(3) 4(1)	3(4) 4(2)	

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