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Cox processes in system degradation modelling

Keywords

stochastic point processes, multiple degradation, Poisson process, stochastic intensity

Abstract

Many systems are subject to multiple degradation processes, which reduce their capability for fulfilling their functions. Pitting corrosion, which consists of the appearance of different that evolve simultaneously in the system, is a classic example of this multiple degradation. It is assumed that the system fails when the pits are large enough that do not allow the system to perform its function, or, in other words, when the degradation level has exceeded a certain fixed threshold, which indicates whether the system is in a good condition. For a system to work properly, a maintenance is performed on it: periodic inspections, repairs, and replacements of the components. The search of the optimal maintenance strategy is a key challenge since we must bear in mind the different costs associated to each maintenance task and the different stochastic processes influencing the system condition: the arrival processes and the growth processes. In this work, we study a system subject to different degradation processes, in which the arrival of those processes is modelled using Cox processes, which are generalizations of the non-homogeneous Poisson process. Using their properties, the survival function, the expected number of arrivals and the expected intensity are obtained.

1. Introduction

Some phenomena that occur repeatedly are mathematically modeled with a point process. However, there are different approaches in the study of these type of processes. The most common theory is the one associated with the renewal processes. The study of the time between failures started in the 20th century with William Feller and during the World War II, when the theory of stochastic processes underwent a spectacular development, extended to other areas like ecology, astronomy, sociology, etc.

Another usual approach is the count of the number of arrivals or events in an interval. Probability theory became more popular thanks to Siméon Denis Poisson and the discover of the Poisson distribution as limit of a binomial distribution.

The counting problem started with the study of the number of storms, blood cells and telephone calls in a telephone switchboard. At the same time, (Cox, 1955) published an article about point processes with different mathematical models, including the Cox or doubly stochastically process (Grandell, 1976). Its novelty was the stochastic nature of the intensity, which was a stochastic process itself. This process is useful to model signal telecommunications, nerve impulses or image reconstruction.

Ogata in 1998 used Cox processes to model the count of earthquakes in a zone, proposing a new algorithm to model these processes with real data

Safety and Reliability of Systems and Processes, Summer Safety and Reliability Seminar 2021.

DOI: 10.26408/srsp-2021-01.

(Oakes, 1975).
Many authors have studied Cox processes from different points of view, highlighting Snyder and Miller (1991), which studied the evolution of the Poisson process to the Cox process, going through the compound Poisson process, displaced process, etc.
Systems are often subject to multiple degradation processes since they can deteriorate in different ways. The simplest example of multiple degradation is the pitting corrosion process. It consists of the appearance of small pits over some metals or other surfaces. When a pit starts, its growth will depend on the environment characteristics, but it has a stochastic nature. Some other examples of this multiple degradation can be found in electronic components or systems subject to fatigue cracking.
Early works on multiple degradation processes assume that there is independence between the different processes. However, this may give a poor and simpler model because degradation processes use to have influence on the subsequent processes and on the system.
Moreover, it is not probably that all degradation processes appear at the same time. They usually initiate at random times, for example, following a non-homogeneous Poisson process. Their growth process can also, of course, be different.
So, we have a combination of two stochastic processes: the initiation process and the growth process.
Dealing with real systems is a key challenge in the sense that the scientific community needs to develop suitable methods which include multidisciplinary aspects. It depends on the assumptions concerning distributions, parameters, or the limited information available (IFR) that we made and requires additional research effort. The decision making in the design of a complex system is based in industrial practice and risk indicators being relevant. The proactive approaching for the management of the lifetime cycle, apart from other new research issues (security and safety of the management process, dependency regarding cost effective maintenance, networks for improved system performance and advanced control and diagnostic systems for limiting errors…) have been developing by the Industry 4.0 solutions. The system modeling here is analyzed from a probabilistic perspective, considering the stochastic processes that determine the arrivals and growth of new degradation processes. However, we could make other assumptions regarding failure distributions, which will be proposed for further studies.
The objectives of this work are the following:

- describe the main concepts of count processes: arrival and interarrival times,
- study the theory of stochastic processes, especially homogeneous and non-homogeneous Poisson processes,
- characterize the Poisson process and the Cox process created from it,
- simulate some Cox processes using the ***thinning algorithm*** programmed with the statistical software ***R***,
- develop mathematical formulas for the expected number of arrivals, theoretically and using simulation,
- give some numerical examples of the maintenance strategy proposed with Cox arrivals.

The chapter is organized into 7 parts: this Introduction as Section 1, Sections 2–6 and Conclusion as Section 7. Some general results on stochastic processes are described in Section 2. Section 3 deals with Cox processes and the shot noise process. Section 4 is devoted to the maintenance policy and Section 5 presents the objective cost function and its optimization with a numerical example.
Finally, the evaluation of the results is discussed in Sections 6 and 7. Wider applications in the field considered in this chapter is suggested in conclusions.

2. Theory of stochastic processes

The theory of stochastic processes provides an analytical framework for modelling the impact of some degradation processes. In maintenance, the most useful processes are those which describe continuous deterioration, such the gamma process, and the Poisson process, which is used for modelling the arrivals of new degradation processes at the system. We are going to study a generalization of the Poisson process, the Cox process, to analyze the intensity of the new arrivals at the system.

2.1. Counting processes

A counting process N_t is a collection of non-negative, integer-valued random variables such that if $0 < s < t$, then $N_s < N_t$. In other words, N_t

counts the number of events up to time t:

$$N_t = \sum \mathbf{1}_{t>T_i}, \quad (1)$$

where $\mathbf{1}$ is the indicator function which takes value 1 when the condition is true and 0 otherwise, and $T_i, i = 1, 2, ...$ are the arrival times of the events at the system.
Supposing $N_0 = 0$, N_t is piecewise constant and has jumps size of 1.

2.2. Homogeneous and non-homogeneous Poisson processes

A Poisson process is a special type of counting process. It is used to model the arrival of events over a continuous interval of time, for example, arrivals of text messages on the phone, accidents occurring in the highway or number of births in a hospital. Also, it might be a good model for earthquakes and natural disasters.
In general, it is widely used to count the occurrences of certain events that appear to happen at a certain rate.

Definition 1. A counting process $(N_t)t$ is a Poisson process with parameter λ that fulfills the following properties:

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with parameter λt.
3. For all $s, t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t.
4. For all $0 < q < r < s < t$, $N_t - N_s$ and $N_r - N_q$ are independent random variables.

Property (3) given in Definition 1 says that the distribution of the number of events in an interval only depends on the length of the interval, and property (4) given in Definition 1 means that the number of events on disjoint intervals are independent random intervals.
To characterize the Poisson process, we are going to focus on the time between the different events instead of the number of events that occur in a fixed interval of time, which is the usual interpretation. With that, we can describe the probabilistic behavior of a stochastic process.
An important concept in Poisson processes is the difference between *arrival* and *interarrival times*. Let X denote the time of the first arrival. With that, $X > t$ if and only if there are no events in the interval $[0, t]$. Therefore, the probability of observing an event at time $X > t$ is:

$$P(X > t) = P(N_t = 0) = \exp(-\lambda t).$$

Computing the probability of observing an interarrival time longer than a fixed time length t, we have:

$$F(t) = P(X < t) = 1 - \exp(-\lambda t).$$

In fact, X has an *exponential distribution* with parameter λ.

Definition 2 (Memoryless property). A random variable is said to be memoryless if for all $s, t > 0$ then,

$$P(X > s + t | X > s) = P(X > t).$$

Lemma 1. The exponential distribution (which appears in a Poisson process) has the memoryless property.
This distribution plays an important role in Poisson processes. Since interarrival times are independent and identically distributed, as we have seen in properties (3) and (4), the previous distribution is true for all interarrival times.
In general, we call $S_1, S_2,\ldots$ the arrival times of the process, and $X_1, X_2,\ldots$ the interarrival times, where:

$$X_k = S_k - S_{k-1},$$

that is, X_k is the time between the $(k-1)-th$ and the $k-th$ arrival.
The interarrival times are independent and each of them follow an exponential distribution with parameter λ.
The previous model is called the homogeneous Poisson process. However, describing more complex processes could be tedious, so we need able to vary the event intensity in time, that is, the rate of the event arrivals is a function of time, $\lambda = \lambda(t)$. Using that intensity, we have the non-homogeneous Poisson process. The main difference with the homogeneous Poisson process is property (2) given in Definition 1. In this case, N_t has a Poisson distribution with mean:

$$E(N_t) = \int_0^t \lambda(u) du. \quad (2)$$

A Poisson process can be completely characterized by its conditional intensity function.

2.3. Simulation algorithm for the Poisson processes

Lemma 2. Let $S_1, S_2, \ldots$ be the arrival times of a Poisson process with parameter λ. Conditioning on $N_t = n$, the distribution of $(S_1, S_2, \ldots)$ is the distribution of n independent and identically uniform random variables on $[0, t]$.

The methods for simulating the arrival times of a homogeneous Poisson process and a non-homogeneous Poisson process are shown below.

Simulation procedure (*homogeneous*):

- simulate the number of arrivals n for a Poisson distribution with parameter $\lambda\, t$,
- generate n uniform random variables on $(0, t)$,
- put the variables in increasing order. This vector is the vector of arrival times of a Poisson process.

Simulation procedure (*non-homogeneous*):

- simulate the number of arrivals n for a Poisson distribution with parameter

$$m(t) = \int_0^t \lambda(u) du,$$

- generate n uniform random variables on $(0, t)$, with density $\lambda / m(t)$,
- sort the previous variables as

$$U_{(1)} < U_{(2)} < U_{(3)} < \cdots$$

- for every i, set $T_i = U_{(i)}$.

3. Cox processes

Definition 3. Let ξ be a random measure on X. A point process $N(-)$ on X is a Cox process directed by ξ when, conditioning on ξ, the realizations of $N(-)$are the realizations of a Poisson process $N(-|\xi)$ on X with parameter measure ξ.

A Cox process (or a doubly stochastic Poisson process) is a generalization of a Poisson process, where the intensity is itself a stochastic process.

An important example of a Cox process is the Shot-noise or trigger process.

3.1. Shot noise process

We assume here that the arrival intensities of the degradation processes follow a Cox process driven by a shot-noise process. This means that the system is subject to shots according to a Poisson process with a deterministic rate μ. In addition, the shot-induced stress is additive and decays with time according to a classical exponential function.

The stochastic intensity of the shot-noise process is given by:

$$\lambda^*(t) = \lambda_0 + \sum_{i=1}^{N(t)} \exp\left(-\delta(t - T_i)\right), \qquad (3)$$

where $N(t)$ is the counting process associated with the homogeneous Poisson process, T_i are the arrival times and $\lambda_0, \delta > 0$.

Given the intensity, the expectation can be calculated by conditioning on $N(t) = n$ and using simulations of a random variable U with uniform distribution in $(0, t)$ as:

$$E\left(\lambda^*(t)\right) = \lambda_0 + E\left(\sum_{i=1}^{N(t)} \exp\left(-\delta(t - T_i)\right)\right)$$
$$= \lambda_0 + \frac{\mu}{\delta}(1 - \exp(-\delta\, t)). \qquad (4)$$

With μ being the rate parameter of the initial Poisson process.

Lemma 2. Let $\lambda^*(t)$ be the random intensity function of the counting process $N^*(t)$. Then,

$$E[N(t)] = \int_0^t E[\lambda^*(s)] ds. \qquad (5)$$

Proof. This is a general result for stochastic process and the proof can be found in (Bertoin, 1996) and (Pinsky & Karlin, 2011).

Using Lemma 2, the expected number of degradation processes at time t is given by:

$$E[N^*(t)] = \int_0^t E[\lambda^*(u)] du$$
$$= \lambda_0 t + \frac{\mu t}{\delta} + \frac{\mu}{\delta^2}(\exp(-\delta t) - 1). \qquad (6)$$

Finally, the survival function of the shot-noise process is next computed. Let W_1 be the instant at which, for the first time, the deterioration level of a degradation process exceeds the corrective or failure threshold L,

$$W_1 = \min\{W_i\},\ i = 1,2, \ldots .$$

The survival function of this variable is:

$$\overline{F_{W_1}}(t) = P(N_L(t) = 0)$$
$$= E\left[\exp\left\{\left(-\int_0^t \lambda_L(u) du\right)\right\}\right]$$
$$= E\left[\exp\left\{\left(-\int_0^t \lambda^* F_{\sigma_L}(t - u) du\right)\right\}\right]. \qquad (7)$$

4. System maintenance

We consider a system subject to multiple degradation processes. That is, defects or “failures” appear at random times instead of at the same time. The arrival intensities of the processes follow a Cox process with stochastic intensity depending on the time.

4.1. General assumptions of the model

Supposing that multiple degradation processes arrive to a system, we are in position to study the stochastic process that describes the deterioration growth of the system. A gamma process is used to model this growth, due to its mathematical properties.
The gamma process is considered one of the most appropriated processes for modelling the damage produced by cumulative deterioration of systems. It has independent and non-negative gamma increments with identical scale parameter. Abdel-Hammed (Abdel-Hammed, 1975) proposed it for modelling random deterioration in time. After that, it has been widely used in the maintenance field and to model different real situations related to making decisions in industry, railway tracks, breakwaters, steel pressure levels, rock dumping etc.
The markovian property allows us to simplify the analytic development of these processes. Exponential distribution and gamma processes are appropriate candidates to model the behaviour of a system. Focusing on the degrading, other stochastic processes maintaining the markovian property could be chosen and the development of the kernel would be similar.
Definition 4 (Gamma process). A gamma process with shape $\alpha(t) > 0$ function and scale parameter $\beta > 0$ is a continuous stochastic process with the following properties:

1. $X(0) = 0$ with probability 1.
2. $X(t) - X(s)$ follows a gamma distribution with parameters $\left(\alpha(t) - \alpha(s)\right)$ and β.
3. $X(t)$ has independent increments.

Its probability density function is given by:

$$f_{\alpha,\beta} = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x),$$

where

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} \exp(-t) dt.$$

The random variable $\sigma_L(t)$ is known as the first hitting time distribution and has the following cumulative distribution function:

$$F_{\sigma_L(t)} = P(X(t) \geq L) = \int_L^{\infty} f_{\alpha t,\beta}(x) dx \frac{\Gamma(\alpha t, \beta L)}{\Gamma(\alpha t)},$$

for $t > 0$, where $f_{\alpha t,\beta}$ is given by Definition 4 and

$$\Gamma(\alpha, x) = \int_x^{\infty} z^{\alpha-1} \exp(-z)\, dz,$$

denotes the incomplete gamma function.
This distribution function indicates the time when the gamma deterioration process exceeds a corrective threshold, denoted by the variable L.

4.2. Maintenance strategy

A self-announcing failure policy means that, whenever a component or a system fails, a warning alarm is triggered. Despite this failure, if possible, the system continues working until the maintenance team arrival. To avoid additional costs, we assume that the degradation level of a system cannot be directly observed, so that the system is periodically inspected each certain time. This is called periodical monitoring, and it is less expensive than a continuous monitoring of the system using sensors. At each periodic inspection, the level of deterioration of the system is observed with the naked eye, having previously established certain damage thresholds. If this level exceeds any of them, the corresponding maintenance of the system is carried out, which consists of being completely replaced, leading to a level 0 of degradation.
These periodic inspections will describe the exact state of the system in those times.
To sum up, we have the following assumptions on the system.

- The system is inspected each T time units. In the inspections, the deterioration level of the system is checked. We have different scenarios:
  - if the degradation level of one of the degradation processes exceeds the preventive threshold, but the system is not failed, a preventive replacement of the corresponding component is performed,
  - if the degradation level of at least one degradation process exceeds the corrective

threshold, that is, the system is failed, then a corrective replacement of the corresponding component is performed,
  - the system is left as it is if there is no degradation process that exceeds the preventive threshold.
- To study the maintenance strategy, a sequence of costs is imposed to the different maintenance actions:
  - corrective replacement cost: C_c monetary units,
  - preventive replacement cost: C_p monetary units,
  - inspection cost: C_i monetary unit,
  - downtime cost: C_d monetary units.

Obviously, the cost due to corrective replacements is always greater than the cost due to preventive replacements.

5. The objective function

Next, the objective cost function is shown in this section. The corresponding survival function is computed both theoretically and with simulations. On the other hand, the expectation of the stochastic intensity of the process and the expected number of arrivals in a period are also calculated.
From a practical point of view, the deterioration of the system may incur high costs, so that is way maintenance plays an important role in most companies.
We consider a shot-noise process, which is a particular Cox process. By (Caballé et al., 2015) and (Castro et al., 2015), the expected cost rate for this maintenance model is given by:

$$C(T,M) = \frac{E[C]}{E[R]}$$

where $E[C]$ is the expected cost in a replacement cycle and $E[R]$ is the expected time to a replacement. Therefore, developing that expression, we have:

$$C(T,M) = \frac{C_c \sum_{k=1}^{\infty} P_c(kT) + C_p \sum_{k=1}^{\infty} P_p(kT)}{E[R]} + \frac{C_I E[N_I] + \sum_{k=1}^{\infty} C_d\big((k+1)T\big)}{E[R]}, \quad (8)$$

where C_c, C_p, C_I and C_d are the corresponding costs due to a maintenance action and $P_c(kT), P_p(kT)$ and $E[N_I]$ represent the probability of a corrective replacement in a time instant kT, the probability of a preventive replacement in kT and the expected number of inspections in the system, respectively.

5.1. A numerical example

We are studying here a realization of a shot-noise Cox process and how to find and evaluate the cost of the maintenance strategy described previously. Consider the sequence of values shown in Table 1 for the different maintenance actions. The values for the gamma deterioration process and the parameters of the shot-noise Cox process are also given in Table 2.

Table 1. Values for the different costs of the maintenance tasks

Cost of the maintenance task	Value
Corrective replacement C_c	100 monetary units
Preventive replacement C_p	50 monetary units
Cost of a periodic inspection C_i	30 monetary units
Cost due to downtimes of the system C_d	20 monetary units per time unit

Table 2. Parameter values for the gamma process and the Cox process

Parameters for the gamma process and the Cox process	Value
Gamma shape parameter α	1.5
Gamma scale parameter β	2
Parameter λ_0	1
Parameter δ	0.5
Poisson process rate μ	1.5
Failure threshold of the gamma process L	8

A cost due to shut down is included, which generates economic losses, since the system is no working during this certain period.

The costs are specific for this numerical example, and logical in the sense that the cost due to corrective replacement is higher than the cost due to preventive replacement. On the other hand, the cost due to periodic inspection is also lower than the cost due to a replacement (corrective or not).

The total expected cost for this maintenance strategy is 35.7049 monetary units per unit time, with optimal values for $T = 6.36$ and $M = 6.11$. All the simulations were made with the statistical software *R* and using a Genetic Algorithm combined with Monte-Carlo method.

The parameters for the gamma process shown in Table 2, especially the value of the shape parameter, mean that the process is increasing failure rate (IFR), that is, the failure rate increases with time and the probability of a failure to happen is higher. This fact encourages us to employ a preventive maintenance policy.

Moreover, the survival function for a realization of a shot-Cox noise process until 50 units of time is represented in Figure 1. It has the usual form: initially there is a high probability of survival, but this decreases dramatically after 15 units of time.

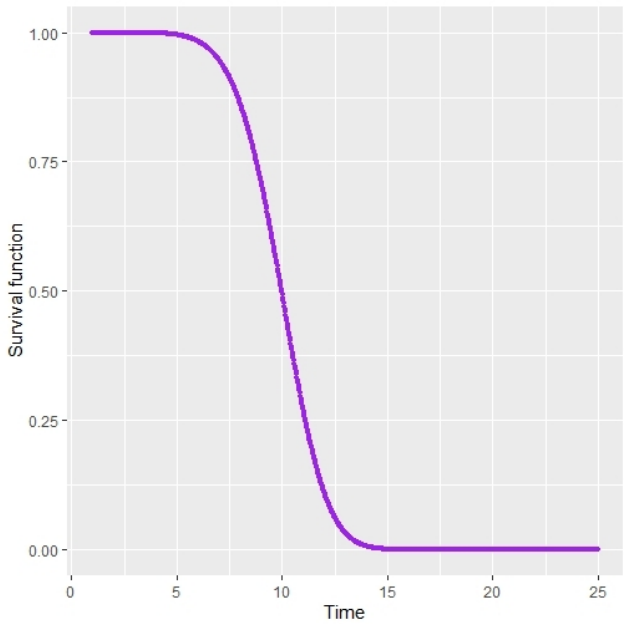

Figure 1. Survival probability until $t = 50$.

Figure 2 shows a representation of the steadily increasing intensity function of the shot Cox-noise process. Note that it stabilizes at value 4 approximately. The parameter of the shot is $\delta = 0.5$, so that the growth rate is not as fast.

As expected, since the degradation processes initiation times follow a shot-Cox noise process, the intensity function of the process is always increasing, and the frequency of the arrivals is higher as time goes by.

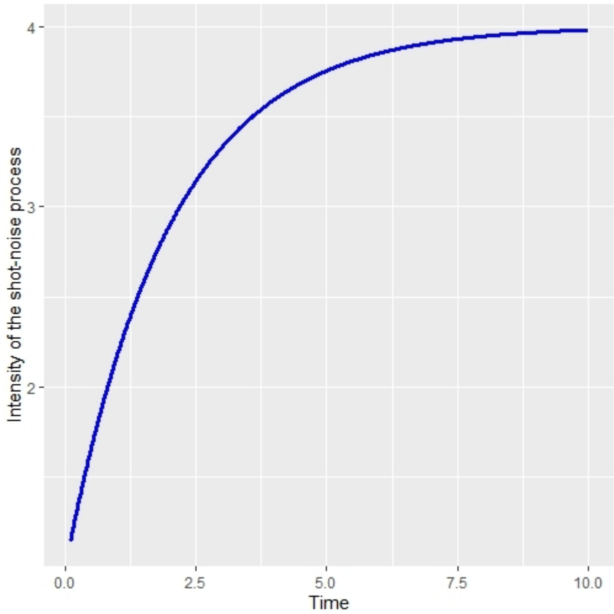

Figure 2. Intensity of the shot noise until $t = 50$.

Figure 3 shows the expected number of arrivals of the shot-Cox noise process until 50 units of time. It is observed to be linear over time, the deterioration processes arrive at the system following a lineal relation.

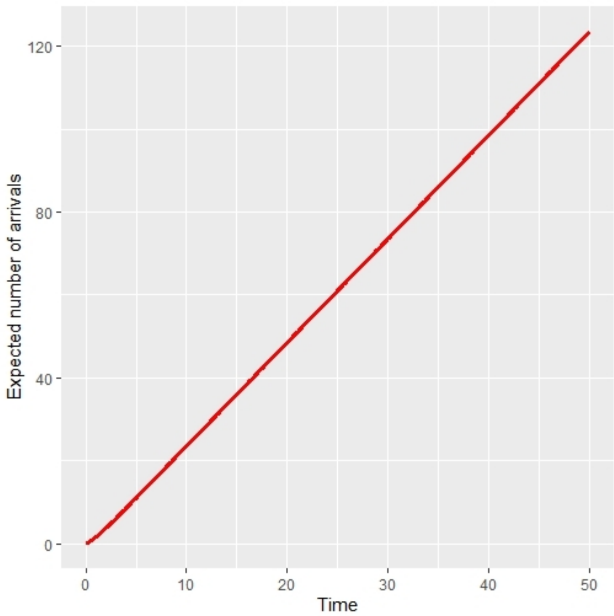

Figure 3. Expected number of arrivals until $t = 50$.

A shot-Cox noise arrival process until time $t = 10$ with the same previous parameters is represented in Figure 4.

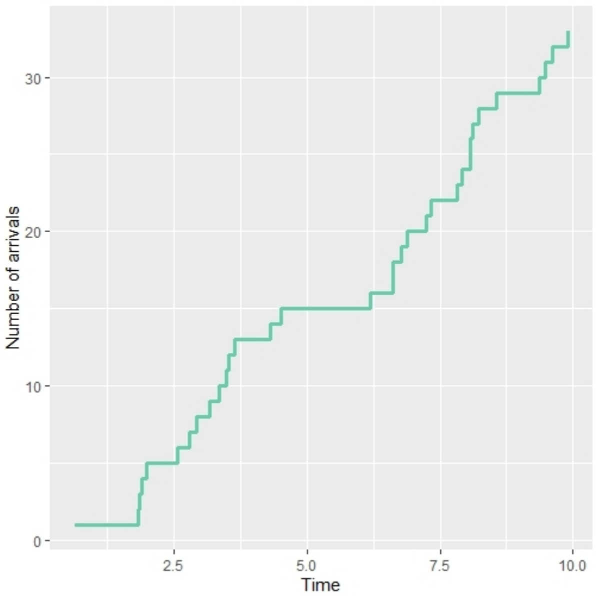

Figure 4. Number of arrivals for t = 10.

A brief sensitivity analysis of the shape and scale parameters' choice of the gamma process is performed to study their influence on the model. In Table 3, the expected cost for each pair of values is calculated, always in monetary units per unit time. It is observed that, as α increases, the expected total cost of maintenance also increases, due to the higher frequency of failures. The same is true as β increases. However, it can be seen that the highest cost is around β = 1.45, compared to the rest of the values taken. Cost varies little at very close values. The differences are found as we move away from the initial value of the chosen parameter.

Table 3. Sensitivity analysis of the gamma process parameters

α \ β	1	1.15	1.3	1.45	1.6
1	18.35	20.21	21.05	25.12	20.34
1.15	21.27	23.56	23.14	27.59	23.51
1.3	23.45	26.02	26.84	31.77	24.48
1.45	25.98	28.94	27.44	33.98	25.11
1.6	27.12	30.46	30.95	38.24	29.41

6. Limitations of the work

The main limitation of the work is the lack of application of the model to a real data set. An attempt has been made to use parameters as close as possible to a hypothetical model in real life, so that we have a model with increasing failure rate, continuous degradation, and periodic inspections of the state of the system.

Regarding the frequency of inspections or the continuous monitoring, we have chosen periodic inspections for their simplicity and low cost, in order not to complicate the model from a probabilistic point of view, knowing that nowadays we have much more sophisticated mechanisms to carry them out.

The technology available, for example, in piping and electrical systems, information technology systems (IT), and, in general, operational technology including industrial automation and control systems (IACS), is much more useful for online diagnostics.

7. Conclusion

A model combining arrivals and growth of stochastic processes is explained in this work. The growth is modelled through the well-known gamma deterioration process, while the arrivals of new processes follow a shot-Cox noise process, which is the main novelty. The shot noise process is described in detail and a numerical example using it is provided. Optimizing the maintenance strategy, that is, finding the optimal values for the time between inspections and the preventive threshold, we can develop a useful model for minimizing the costs of a system maintenance.

Regarding to the obtained values, we are in position to say that the time between inspections in a model with our parameters should be approximately 6.36 units of time, and the preventive threshold should be set at 6.11. This model is suitable to represent the shocks in a system produced by external agents: climate, shocks, deterioration etc. Each shock increases the intensity of the process, so they will be more and more frequent. This is the main novelty of the shot noise process, its continuously increasing intensity, which will produce a greater number of shocks.

Future work is focus on study some self-exciting processes in maintenance theory. They are point processes in which the arrival of an event causes the intensity function to increase. The first self-exciting process was proposed by (Hawkes, 1971). In that, the intensity function depends on the previous events, not on another external process (the Poisson process), as is the case in shot noise. It is defined with the following stochastic intensity function:

$$\lambda^*(t) = \lambda + \int_0^t \mu(t-u)dN(u).$$

For some $\lambda > 0$ and $\mu: (0,\infty) \to [0,\infty)$ which are called the background intensity and the excitation function, respectively. If $\mu = 0$, we have the trivial case of a homogeneous Poisson process.
In these self-exciting processes, the Ogata-Oakes algorithm (Oakes, 1975) will be used for their simulation.

Acknowledgment

This research was supported by Junta de Extremadura (Projects GR18108, DEEP-EM (Ref IB18073) and GR18055), the Spanish Government and FEDER by projects TEC2017-85376-C2-1-R and MTM2015-63978-P, and European Union (European Regional Development Funds). The authors would like to thank the anonymous reviewers for their thoughtful comments and efforts towards improving our manuscript.

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