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Safety analysis of multistate ageing car wheel system with dependent components

Keywords

multistate system, ageing, dependent failures, safety, car transport

Abstract

An innovative approach and a new significant theoretical result are proposed for the safety analysis of multistate ageing systems that consider their components' dependency. A safety function and a risk function are defined and determined for a multistate ageing system with independent and dependent components. As a special case, the safety of a series system is modelled using its components' piecewise exponential safety functions. Results are applied to examine and characterize safety of an exemplary car wheel system.

1. Introduction

The approach to multistate ageing system reliability analysis was introduced in [15]–[17] and widely developed and transferred to safety analysis in [4], [7]–[10], [12]–[14]. Next, the approach was developed to reliability analysis of systems and critical infrastructure networks composed of dependent components and subsystems in [1]–[2]. Those practically important approaches to multistate system reliability and safety analysis consider the assumption about component degradation through departures from the reliability state subsets instead of component failures. It is a natural assumption as in real technical systems, components often degrade with time by going to states corresponding to different performance levels before they fail completely. Degradation of components and subsystems in case of complex systems, causes the decreasing of system reliability and its operation safety. The paper is devoted to joining of a multistate system ageing and its components inside dependences to consider them together in system safety analysis and to show the possibility of its real application in practice. The paper is organized into 5 parts, this Introduction as Section 1, Sections 2–4 and Summary as Section 5. Section 2 is devoted to introducing the multistate approach to ageing system safety analysis. In Section 3, the safety of a multistate ageing system without inside dependences and outside impacts is discussed and modelled. The safety function of a series system and its components are defined and determined. A possibility of real technical application of the proposed safety model to the car wheel system without inside dependences safety examination is illustrated and its safety indicators are evaluated. In Section 4, the safety of multistate ageing system impacted by its components inside dependency is considered. The system components dependency rule is introduced and new theorem on safety of homogeneous ageing series system impacted by its components dependency according to this rule and its particular case in the form of corollary when the system components have piecewise safety functions are formulated and justified. The safety of car wheel system impacted by its components dependency described by the assumed dependency rule is examined and its safety indicators are determined. In Summary, the results' evaluation and the possibility of their real practical applications are discussed and the perspective for future research in the field considered in the paper is delineated.

2. Multistate approach to ageing system safety

Similarly, as in the case of multistate approach to system reliability [8], in the multistate system safety analysis to define the system with degrading/ageing components, we assume that [9]–[10]:

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- n is the number of the system components;
- E_i, $i = 1,2,...,n$, are the system components;
- all components and the system have the safety state set $\{0,1,...,z\}$, $z \geq 1$;
- the safety states are ordered, the safety state 0 is the worst and the safety state z is the best;
- r, $r \in \{1,2,...,z\}$, is the critical safety state (the system and its components staying in the safety states less than the critical state, i.e. in safety states 1, 2, …, r – 1, is highly dangerous for them and for their operating environment);
- $T_i(u)$, $i = 1,2,...,n$, are random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, $u = 0, 1,2,...,z$, while they were in the safety state z at the moment $t = 0$;
- $T(u)$ is a random variable representing the lifetime of the system in the safety state subset $\{u,u+1,...,z\}$, $u = 0,1,2,...,z$, while it was in the safety state z at the moment $t = 0$;
- the safety states degrade with time t;
- the components and the system degrade with time t;
- $s_i(t)$ is the component E_i, $i = 1,2,...,n$, safety state at the moment t, $t \in < 0,\infty)$, while it was in the safety state z at the moment $t = 0$;
- $s(t)$ is the system safety state at the moment t, $t \in < 0,\infty)$, given that it was in the safety state z at the moment $t = 0$.

The above assumptions mean that the safety states of the system with degrading components may be changed in time only from better to worse [8]–[10].

We define the system safety function by the vector [8]–[10]

$$\boldsymbol{S}(t,\cdot) = [\, \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2), ...,\ \boldsymbol{S}(t,z)\,],\ t \in < 0,\infty), \quad (1)$$

where

$$\boldsymbol{S}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t),$$
$$t \in < 0,\infty),\ u = 1,2,...,z, \quad (2)$$

is the probability that the multistate system is in the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,...,z$, at the moment t, $t \in < 0,\infty)$, while it was in the safety state z at the moment $t = 0$.

We do not consider in the vector (1) the function $\boldsymbol{S}(t,0)$ as

$$\boldsymbol{S}(t,0) = P(s(t) \geq 0 \mid s(0) = z) = P(T(0) > t) = 1,$$

for $t \in < 0,\infty)$, what means that it is constant. The safety functions $\boldsymbol{S}(t,u)$, $t \in < 0,\infty)$, $u = 1,2,...,z$, defined by (2) are called the coordinates of the system safety function $\boldsymbol{S}(t,\cdot)$, $t \in < 0,\infty)$, given by (1). Thus, the relationship between the distribution function $\boldsymbol{F}(t,u)$, of the system lifetime $T(u)$, $u = 1,2,...,z$, in the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,...,z$, and the coordinate $\boldsymbol{S}(t,u)$, $t \in < 0,\infty)$, $u = 1,2,...,z$, of its safety function is given by

$$\boldsymbol{F}(t,u) = P(T(u) \leq t) = 1 - P(T(u) > t)$$
$$= 1 - \boldsymbol{S}(t,u),\ t \in < 0,\infty),\ u = 1, 2, ..., z.$$

The exemplary graph of a five-state ($z = 4$) system safety function

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3), \boldsymbol{S}(t,4)],\ t \in < 0,\infty),$$

is shown in *Figure 1*.

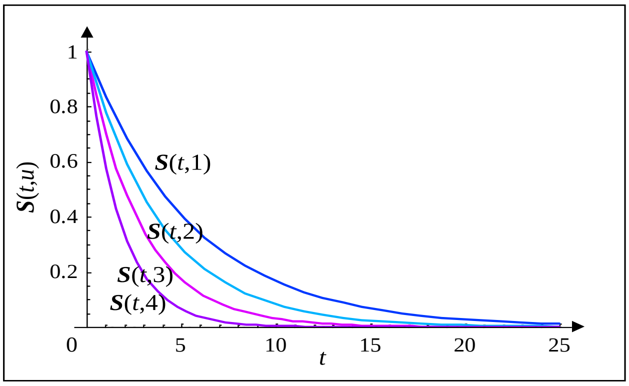

Figure 1. The graphs of a five-state system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

If r is the critical safety state, then the multistate system risk function

$$\boldsymbol{r}(t) = P(s(t) < r | s(0) = z) = P(T(r) \leq t),$$
$$t \in < 0,\infty), \quad (3)$$

is defined as a probability that the system is in the subset of safety states worse than the critical safety state r, $r \in \{1,2,...,z\}$, while it was in the best safety state z at the moment $t = 0$ and given by [8]–[10]

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,r),\ t \in < 0,\infty), \quad (4)$$

where $\boldsymbol{S}(t,r)$ is the coordinate of the multistate system safety function (1) given by (2) for $u = r$.

The graph of the exemplary system risk function is presented in *Figure 2*.

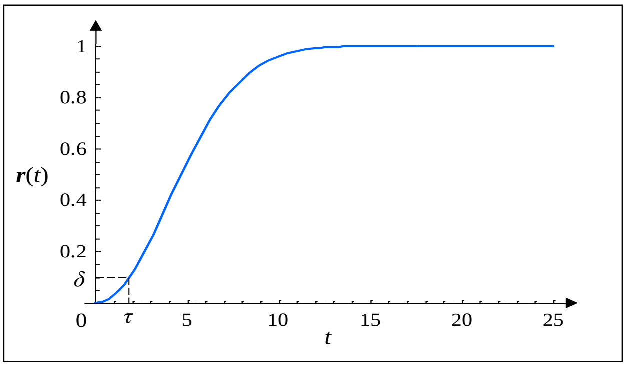

Figure 2. The graph of the exemplary system risk function $\boldsymbol{r}(t)$

3. Safety of multistate ageing system without inside dependences

3.1. Multistate ageing system with independent components

In addition to the agreements of Section 2, we assume that the system components E_i, $i = 1,2,...,n$, are independent, Further, we denote by $T_i^0(u)$, $u = 0,1,2,...,z$, $i = 1,2,...,n$, the independent random variables representing the lifetimes of components E_i, $i = 1,2,...,n$, in the safety state subset $\{u,u+1,...,z\}$, $u = 0,1,2,...,z$, while they were in the safety state z at the moment $t = 0$ and by $T^0(u)$, $u = 0,1,2,...,z$, the random variable representing the lifetime of the system composed of these independent components in the safety state subset $\{u,u+1,...,z\}$, $u = 0,1,2,...,z$, while it was in the safety state z at the moment $t = 0$.
We define the system independent components E_i, $i = 1,2,...,n$, safety functions by the vectors [8]–[10]

$$S_i^0(t,\cdot) = [\, S_i^0(t,1),\ S_i^0(t,2),\ \ldots, S_i^0(t,z)\,],$$
$$t \in <0,\infty),\ i = 1,2,...,n, \qquad (5)$$

with the coordinates

$$S_i^0(t,u) = P([s_i(t) \geq u | s_i(0) = z) = P(T_i^0(u) > t),$$
$$t \in <0,\infty),\ u = 1,2,...,z,\ i = 1,2,...,n. \qquad (6)$$

The functions $S_i^0(t,0)$, $i = 1,2,...,n$, are not included in the vector (5) because

$$S_i^0(t,0) = P(s_i(t) \geq 0 \mid s_i(0) = z) = P(T_i(0) > t) = 1,$$

for $t \in <0,\infty)$, $i = 1,2,...,n$, i.e. they are constant.

The safety function (5) coordinate $S_i^0(t,u)$, $t \in <0,\infty)$, $u = 1,2,...,z$, $i = 1,2,...,n$, defined by (6), is the probability that the component E_i, $i = 1,2,...,n$, lifetime $T_i^0(u)$, $u = 1,2,...,z$, in the safety state subset $\{u,u+1,...,z\}$, is greater than t.
Similarly, we define the safety function of the system of independent components E_i, $i = 1,2,...,n$, by the vector [8]–[10]

$$\boldsymbol{S}^0(t,\cdot)] = [\, \boldsymbol{S}^0(t,1),\ \boldsymbol{S}^0(t,2),\ ...,\ \boldsymbol{S}^0(t,z)\,],$$
$$t \in <0,\infty), \qquad (7)$$

with the coordinates

$$\boldsymbol{S}^0(t,u) = P([s(t) \geq u | s(0) = z) = P(T^0(u) > t),$$
$$t \in <0,\infty),\ u = 1,2,...,z. \qquad (8)$$

The safety function (7) coordinate $\boldsymbol{S}^0(t,u)$, $t \in <0,\infty)$, $u = 1,2,...,z$, defined by (8) is the conditional probability that the multistate ageing system composed of independent components lifetime $T^0(u)$, $u = 1,2,...,z$, in the safety state subset $\{u,u+1,...,z\}$, $u = 1,2,...,z$, is greater than t.

3.2. Multistate ageing series system with independent components

On the basis of the multistate approach to the system safety analysis in Section 2, considering the definition of the reliability function of multistate system and its reliability structure introduced in [8], we may similarly define the multistate system safety structures. The simplest multistate system safety structure is a series safety structure defined as follows.

Definition 1. A multistate system composed of n independent components E_i, $i = 1,2,...,n$, is called series if its lifetime $T_i^0(u)$, $u = 1,2,...,z$, $i = 1,2,...,n$, in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,...,z$, is given by

$$T^0(u) = \min_{1 \leq i \leq n} \{T_i^0(u)\},\ u = 1,2,...,z,$$

where $T_i^0(u)$, $u = 0,1,2,...,z$, $i = 1,2,...,n$, are the independent random variables representing lifetimes of the independent components E_i, $i = 1,2,...,n$, in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,...,z$. The number n is called the system safety structure shape parameter.
The above definition means that the series multistate system is in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,...,z$, if and only if all its n components are in this subset of safety states. Thus, the meaning of this definition is very close to the definition

of a two-state system considered in a classical reliability analysis that is not failed if all its components are not failed [8]. This fact can justify the safety structure scheme for a series system presented in *Figure 3*.

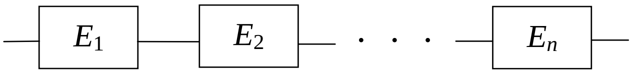

Figure 3. The scheme of a series multistate system safety structure

It is easy to work out that the safety function of the series multistate ageing system composed of independent components is given by the vector [9]–[10]

$$[\boldsymbol{S}^0(t,\cdot)] = [\ \boldsymbol{S}^0(t,1),\ \boldsymbol{S}^0(t,2), ..., \boldsymbol{S}^0(t,z)\ ],$$
$$t \in <0,\infty), \qquad (9)$$

with the coordinates

$$\boldsymbol{S}^0(t,u) = \prod_{i=1}^{n} S_i^0(t,u),\ \ t \in <0,\infty), u = 1,2,...,z, \qquad (10)$$

where

$$S_i^0(t,u) = P(s_i(t) \geq u \mid s_i(0) = z) = P(T_i^0(u) > 0),$$
$$t \in <0,\infty),\ \ u = 1,2,...,z,\ \ i = 1,2,...,n, \qquad (11)$$

are the coordinates of the component E_i, $i = 1,2,...,n$, safety function defined by the vector

$$S_i^0(t,\cdot) = [\ S_i^0(t,1),\ S_i^0(t,2), \dots, S_i^0(t,z)\ ],$$
$$t \in <0,\infty),\ \ i = 1,2,...,n. \qquad (12)$$

In the case, the series system is homogeneous with the components having the same safety functions, i.e.

$$S_i^0(t,\cdot)\ =\ S^0(t,\cdot)\ = [\ S^0(t,1),\ S^0(t,2), \dots, S^0(t,z)\ ],$$
$$t \in <0,\infty),\ \ i = 1,2,...,n, \qquad (13)$$

where

$$S^0(t,u) = S_i^0(t,u) = P(s_i(t) \geq u | s_i(0) = z)$$
$$= P(T_i\,(u) > t),\ \ t \in <0,\infty),$$
$$u = 1,2,...,z,\ \ i = 1,2,...,n, \qquad (14)$$

the formula (10) for the coordinates of the series multistate system safety function (9) takes the following form

$$\boldsymbol{S}^0(t,u) = [S^0(t,u)]^n,\ \ t \in <0,\infty),\ \ u = 1,2,...,z. \qquad (15)$$

Safety structures of other system composed of independent components safety structures can be defined in an analogous way and the formulae for their safety functions can be find in [8].

3.3. Functional and safety structures of car wheel system

We consider the car wheel system composed of four components:

- E_1 – a left front wheel of the car,
- E_2 – a right front wheel of the car,
- E_3 – a left rear wheel of the car,
- E_4 – a right rear wheel of the car.

The functional structure of the car wheel system is presented in *Figure 4*.

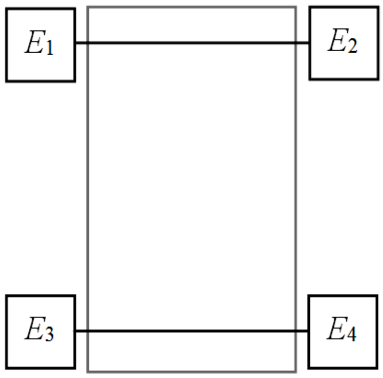

Figure 4. The functional structure of the car wheel system

The considered car wheel system is a series system with the safety structure presented in Figure *5*.

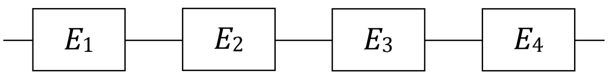

Figure 5. The safety structure of the car wheel system

3.4. Safety parameters of car wheel system with independent components

After considering the comments and opinions coming from experts, taking into account the degradation and safety aspects of the car wheel system and its components operation, we arbitrarily fix for all of them their following parameters:

- the number of safety states (excluding safety state 0) $z = 4$;
- five ($z + 1 = 5$) safety states:
  – a safety state 4 – a component and the car wheel system operation is fully safe,

– a safety state 3 – a component and the car wheel system operation is less safe and dangerous because of the wheel tire wear,
– a safety state 2 – a component and the car wheel system operation is much less safe and more dangerous because of the wheel tire and its mechanical parts wear,
– a safety state 1 – a component and the car wheel system operation is much less safe and more dangerous because of the significant wheel tire and its mechanical parts wear,
– a safety state 0 – a component and the car wheel system operation is unsafe because of the wheel failure;

and we assume that:

- there are possible the transitions between the components and the car wheel system safety states only from better to worse ones;
- the critical safety state of the components and the car wheel system is $r = 2$;
- the car wheel system risk function permitted level is $\delta = 0.05$.

Under the above assumptions, we will perform detailed safety analysis of the car wheel multistate system free of inside dependences assuming that its components E_i, $i = 1,2,3,4$, have the same piecewise exponential safety functions of the form

$$S_i^0(t,\cdot) = S^0(t,\cdot) = [S^0(t,1), S^0(t,2), S^0(t,3), S^0(t,4)],$$
$$t \in < 0,\infty), \ i = 1,2,3,4, \qquad (16)$$

with the coordinates

$$S^0(t,u) = S^0_i(t,u) = \exp[-\lambda_i^0(u)t] = \exp[-\lambda^0(u)t],$$
$$t \in <0,\infty), \ u = 1,2,3,4, \ i = 1,2,3,4, \qquad (17)$$

where

$$\lambda_i^0(u), \ u = 1,2,3,4, \ i = 1,2,3,4,$$

are the intensities of ageing of the system components E_i, $i = 1,2,3,4$, also called the intensities of departure from the safety state subsets $\{u, u+1, \ldots, 4\}$, $u = 1,2,3,4$.

Further, we arbitrarily assume the following intensities of the system components E_i, $i = 1,2,3,4$, departure from the safety state subsets:

- for the safety state subset {1,2,3,4}

$$\lambda_i^0(1) = \lambda^0(1) = 0.0125,$$

- for the safety state subset {2,3,4}

$$\lambda_i^0(2) = \lambda^0(2) = 0.0200,$$

- for the safety state subset {3,4}

$$\lambda_i^0(3) = \lambda^0(3) = 0.0250,$$

- for the safety state subset {4}

$$\lambda_i^0(4) = \lambda^0(4) = 0.03125, \text{ for } i = 1,2,3,4, \qquad (18)$$

what implies that the coordinates (17) of the system components E_i, $i = 1,2,3,4$, safety functions (16) have the forms:

$$S^0_i(t,1) = S^0(t,1) = \exp[-0.0125t],$$
$$S^0_i(t,2) = S^0(t,2) = \exp[-0.0200t],$$
$$S^0_i(t,3) = S^0(t,3) = \exp[-0.0250t],$$
$$S^0_i(t,4) = S^0(t,4) = \exp[-0.03125t],$$
$$t \in < 0,\infty), \ i = 1,2,3,4. \qquad (19)$$

3.5. Safety indicators of car wheel system with independent components

After applying formulae for the safety function of the homogeneous series multistate system (9)–(15) and considering (16)–(17) and (19), we obtain the safety function of the car wheel system

$$\boldsymbol{S}^0(t,\cdot) = [\boldsymbol{S}^0(t,1), \boldsymbol{S}^0(t,2), \boldsymbol{S}^0(t,3), \boldsymbol{S}^0_i(t,4)],$$
$$t \in < 0,\infty), i = 1,2,3,4, \qquad (20)$$

where

$$\boldsymbol{S}^0(t,1) = [\exp[-0.0125t]]^4 = \exp[-0.050t],$$
$$\boldsymbol{S}^0(t,2) = [\exp[-0.0200t]]^4 = \exp[-0.080t],$$
$$\boldsymbol{S}^0(t,3) = [\exp[-0.02500t]]^4 = \exp[-0.100t],$$
$$\boldsymbol{S}^0(t,4) = [\exp[-0.03125t]]^4 = \exp[-0.125t],$$
$$t \in < 0,\infty). \qquad (21)$$

As the critical safety state is $r = 2$, then by (4) and (21), the car wheel system risk function is

$$\boldsymbol{r}^0(t) = 1 - \boldsymbol{S}^0(t,2) = 1 - \exp[-0.080t],$$
$$t \in < 0,\infty), \qquad (22)$$

and by the formula for the inverse risk function [8], considering (22), the moment of exceeding acceptable value of critical infrastructure risk function level $\delta = 0.05$ is

$$\boldsymbol{\tau}^0 = (\boldsymbol{r}^0)^{-1}(0.05) = -12.5\log(0.95)$$
$$= 0.641 \text{ year}. \qquad (23)$$

The graph of the car wheel system risk function is presented in *Figure 6*.

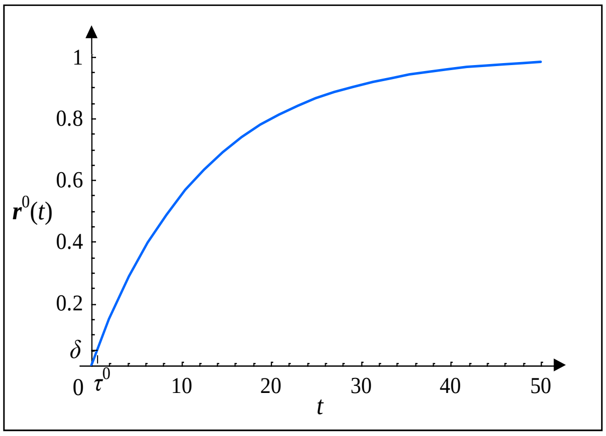

Figure 6. The graph of the car wheel system risk function

Considering (20)–(21) and applying the formulae given in [10], the mean values of the lifetimes of the car wheel system in the safety state subsets are:

- in the safety state subset: {1,2,3,4}

$$\boldsymbol{\mu}^0(1) = \int_0^\infty \boldsymbol{S}^0(t,1)dt = \int_0^\infty \exp[-0.050t]dt$$
$$= 20 \text{ years},$$

- in the safety state subset {2,3,4}

$$\boldsymbol{\mu}^0(2) = \int_0^\infty \boldsymbol{S}^0(t,2)dt = \int_0^\infty \exp[-0.080t]dt$$
$$= 12.50 \text{ years},$$

- in the safety state subset {3,4}

$$\boldsymbol{\mu}^0(3) = \int_0^\infty \boldsymbol{S}^0(t,3)dt = \int_0^\infty \exp[-0.100t]dt$$
$$= 10 \text{ years},$$

- in the safety state subset {4}

$$\boldsymbol{\mu}^0(4) = \int_0^\infty \boldsymbol{S}^0(t,4)dt = \int_0^\infty \exp[-0.125t]dt$$
$$= 8 \text{ years}. \tag{24}$$

From (24), applying formula from [10], the mean lifetimes $\bar{\boldsymbol{\mu}}^0(u)$, $u = 1,2,3,4$, of the car wheel system in the particular safety states $u = 1,2,3,4$, respectively are:

$$\bar{\boldsymbol{\mu}}^0(1) = \boldsymbol{\mu}^0(1) - \boldsymbol{\mu}^0(2) = 7.50 \text{ years},$$
$$\bar{\boldsymbol{\mu}}^0(2) = \boldsymbol{\mu}^0(2) - \boldsymbol{\mu}^0(3) = 2.50 \text{ years},$$
$$\bar{\boldsymbol{\mu}}^0(3) = \boldsymbol{\mu}^0(3) - \boldsymbol{\mu}^0(4) = 2.0 \text{ years},$$
$$\bar{\boldsymbol{\mu}}^0(4) = \boldsymbol{\mu}^0(4) = 8.0 \text{ years}. \tag{25}$$

The intensities of degradation (ageing) of the car wheel system / the intensities of the car wheel system departure from the safety state subsets {1,2,3,4}, {2,3,4}, {3,4}, {4}, i.e.

$$\boldsymbol{\lambda}^0(t,1),\ \boldsymbol{\lambda}^0(t,2),\ \boldsymbol{\lambda}^0(t,3),\ \boldsymbol{\lambda}^0(t,4),\ t \in \langle 0,\infty), \tag{26}$$

can be determined either according to the formula [10]

$$\boldsymbol{\lambda}^0(t,u) = -\frac{d\boldsymbol{S}^0(t,u)}{dt} / \boldsymbol{S}^0(t,u),\ t \in \langle 0,\infty),$$
$$u = 1,2,3,4, \tag{27}$$

where $\boldsymbol{S}^0(t,u)$, $t \in \langle 0,\infty)$, $u = 1,2,3,4$, are given by (21) or by the approximate formula (exact in the case of piecewise exponential system safety function) for the mean intensity

$$\boldsymbol{\lambda}^0(t,u) = \frac{1}{\boldsymbol{\mu}^0(u)},\ t \in \langle 0,\infty),\ u = 1,2,3,4, \tag{28}$$

where $\boldsymbol{\mu}^0(u)$, $u = 1,2,3,4$, are given by (24).
The intensities of degradation defined by (27) and (28) are constant and their values amount:

$$\boldsymbol{\lambda}^0(t,1) = 0.050,\ \boldsymbol{\lambda}^0(t,2) = 0.080,\ \boldsymbol{\lambda}^0(t,3) = 0.100,$$
$$\boldsymbol{\lambda}^0(t,4) = 0.125 \text{ for } t \in \langle 0,\infty). \tag{29}$$

4. Safety of multistate ageing system with inside dependences

4.1. System components dependency rule

We consider an ageing system composed of dependent components E_i, $i = 1,2,\ldots,n$, with safety functions

$$S_i^1(t,\cdot) = [S_i^1(t,1), S_i^1(t,2), \ldots, S_i^1(t,z)],\ \ t \in \langle 0,\infty),$$
$$i = 1,2,\ldots,n, \tag{30}$$

where

$$S_i^1(t,u) = P(T_i^1(u) > t),\ t \in \langle 0,\infty),\ \ u = 1,2,\ldots,z,$$
$$i = 1,2,\ldots,n, \tag{31}$$

and $T_i^1(u)$, $u = 1,2,\ldots,z$, $i = 1,2,\ldots,n$, are the system components E_i, $i = 1,2,\ldots,n$, lifetimes in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, before any

system component leaves this safety state subset. Further, we assume that the system components E_i, $i = 1,2,\ldots,n$, are dependent according to the dependency rule such that if the component E_j, $j = 1,2,\ldots,n$, leaves the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, then the safety parameters of the other components E_i, $i = 1,2,\ldots,n$, $i \neq j$, worsen depending on their exposure to the component E_j safety state change. This influence can be expressed by the coefficients $q(\upsilon, E_j, E_i)$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$ of the component E_j, $j = 1,2,\ldots,n$, impact on other components E_i, $i = 1,2,\ldots,n$, $i \neq j$, lifetimes $T_i^1(u)$, $u = 1,2,\ldots,z$, $i = 1,2,\ldots,n$, $i \neq j$, in the safety state subset $\{u, u+1,\ldots,z\}$, $u = 1,2,\ldots,z$.
Further, we assume that [1]–[2]

$$0 < q(\upsilon, E_j, E_i) \leq 1,\ i = 1,2,\ldots,n,\ i \neq j,$$
$$\text{and } q(\upsilon, E_j, E_j) = 1,\ j = 1,2,\ldots,n. \tag{32}$$

In the considered system components dependency rule, the component lifetimes in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, decrease according to the formula [1]–[2]

$$T_{i/j}^1(\upsilon) = q(\upsilon, E_j, E_i) \cdot T_i^1(\upsilon),\ \upsilon = u, u-1,\ldots,1,$$
$$u = 1,2,\ldots,z-1,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{33}$$

where $T_{i/j}^1(\upsilon)$ denotes the lifetime in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, of component E_i, $i = 1,2,\ldots,n$, after the departure of component E_j, $j = 1,2,\ldots,n$, from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$.
The formula (33) means that, if the component E_j, $j = 1,2,\ldots,n$, leaves the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, then the system components E_i, $i = 1,2,\ldots,n$, lifetimes in a subset of safety states not worse than $\upsilon = u$ i.e. in $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, will decrease, but also their lifetimes in a safety state subset of states not worse than $\upsilon = u-1$ i.e. in $\{u-1,u,\ldots,z\}$, and their lifetimes in a subset of safety states not worse than $\upsilon = u-2$ i.e. in $\{u-2,u-1,\ldots,z\}$, and so on, will be reduced. These subsets of safety states, to which the changes apply, are noted in general by $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u,u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, what is expressed in (33).
The formulae (32)–(33) imply that the mean values $E[T_{i/j}^1(\upsilon)]$, $\upsilon = u,u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, of the system components lifetimes $T_{i/j}^1(\upsilon)$, $\upsilon = u,u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$, decrease according to the formula

$$E[T_{i/j}^1(\upsilon)] = q(\upsilon, E_j, E_i) \cdot E[T_i^1(\upsilon)],\ \upsilon = u, u-1,\ldots,1,$$
$$u = 1,2,\ldots,z-1,\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{34}$$

and the approximate mean intensities of ageing

$$\lambda_{i/j}^1(\upsilon) = 1/E[T_{i/j}^1(\upsilon)],\ \upsilon = u, u-1,\ldots,1,$$
$$u = 1,2,\ldots,z-1, \tag{35}$$

i.e. the approximate mean intensities of departure from the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2,\ldots,z-1$ of the system components increase according to the formula

$$\lambda_{i/j}^1(\upsilon) = \rho_{i/j}^1(\upsilon)\lambda_i^1(\upsilon),\ \upsilon = u, u-1,\ldots,1,$$
$$u = 1,2,\ldots,z-1, \text{ for } i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{36}$$

where

$$\rho_{i/j}^1(\upsilon) = \frac{1}{q(\upsilon, E_j, E_i)},\ \upsilon = u, u-1,\ldots,1,$$
$$u = 1,2,\ldots,z-1, \text{ for } i = 1,2,\ldots,n,\ j = 1,2,\ldots,n, \tag{37}$$

are the coefficients of system components dependency impact on the system components E_i, $i = 1,2,\ldots,n$, approximate mean intensities of aging before the E_j, $j = 1,2,\ldots,n$, departure from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, given by

$$\lambda_i^1(u) = 1/E[T_i^1(u)],\ u = 1,2,\ldots,z,\ i = 1,2,\ldots,n. \tag{38}$$

The formulae (35)–(36) and (38) for mean intensities of ageing are exact if the system components lifetimes $T_i^1(u)$, $u = 1,2,\ldots,z$, in the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, have piecewise exponential distributions.
In particular case, the coefficients $q(\upsilon, E_j, E_i)$ can be functions of the distance d_{ij} between components E_i and E_j, $i,j = 1,2,\ldots,n$, expressed by $q(\upsilon,d_{ij})$, where $d_{ij} = |i - j|$.
In this case, the component lifetimes and their mean values in the safety state subset $\{\upsilon,\upsilon+1,\ldots,z\}$, $\upsilon = u,u-1,\ldots,1$, and $u = 1,2,\ldots,z-1$, decrease according to the formulae [1]–[2]

$$T_{i/j}^1(\upsilon) = q(\upsilon, d_{ij}) \cdot T_i^1(\upsilon),$$
$$E[T_{i/j}^1(\upsilon)] = q(\upsilon, d_{ij}) \cdot E[T_i^1(\upsilon)],\ i = 1,2,\ldots,n,$$
$$j = 1,2,\ldots,n, \tag{39}$$

where the coefficients $q(\upsilon,d_{ij})$, $0 < q(\upsilon,d_{ij}) \leq 1$ and $q(\upsilon,0) = 1$, $\upsilon = u, u-1,\ldots,1$ and $u = 1,2, \ldots, z-1$, are functions of the component distance d_{ij}.
Consequently, we define the safety function of E_i,

$i = 1,\ldots,n$, after the departure of E_j, $j = 1,2,\ldots,n$, from the safety state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, [1]–[2]

$$S^1_{i/j}(t,\cdot) = [S^1_{i/j}(t,1), S^1_{i/j}(t,2),\ldots,S^1_{i/j}(t,z)],$$
$$t \in <0,\infty),\ i = 1,\ldots,n,\ j = 1,\ldots,n, \quad (40)$$

with the coordinates given by

$$S^1_{i/j}(t,\upsilon) = P(T^1_{i/j}(\upsilon) > t),\ t \in <0,\infty),$$
$$\upsilon = u, u-1,\ldots,1,\ u = 1,2,\ldots,z-1, \quad (41)$$

$$S^1_{i/j}(t,\upsilon) = P(T^1_{i/j}(\upsilon) > t) = P(T^1_i(\upsilon) > t) = S^1_i(t,\upsilon),$$
$$\upsilon = u+1,\ldots,z,\ u = 1,2,\ldots,z-1. \quad (42)$$

4.2. Safety of multistate ageing series homogeneous system with dependent components

We consider a homogeneous, ageing series system composed of dependent components, which follow the components dependency rule (34)–(39) fixed in Section 4.1 and have identical safety functions, i.e.

$$S^1_i(t,\cdot) = S^1(t,\cdot) = [S^1(t,1), S^1(t,2),\ldots,S^1(t,z)],$$
$$t \in <0,\infty),\ \ i = 1,\ldots,n. \quad (43)$$

We can notice that at the moment t, $t \in <0,\infty)$, the system is in the safety state subset $\{u,u+1,\ldots,z\}$, if at that moment either all its components are in the subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, (what is expressed in the first part of the formula (45) given below), or any of its components has left the safety state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, at any moment a, $0 < a < t$, and simultaneously, the remaining $n-1$ components have not left the subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, during time a, while the component that has left the state subset $\{u+1,\ldots,z\}$ remains in the safety state subset $\{u,u+1,\ldots,z\}$ during time a, and all system components remain in the safety state subset $\{u,u+1,\ldots,z\}$ during the remaining time $t-a$, $0 < a < t$, with changed safety functions according to the assumed dependency rule what is expressed respectively by subsequent parts of the expression occurring under the integral in (45).

Finally, as a multistate series system is in the best safety state z if and only if all its components are in this safety state, the z-th coordinate of a system safety function takes the same form as for a homogeneous multistate series system with independent components, what is expressed in formula (46). This idea allowed to formulate *Theorem 1* on series system safety presented below that is a slight modification of the proposition proved in [1]–[2] for series system reliability.

Theorem 1.
If, in a homogeneous aging multistate series system with dependent components following the dependency rule (34)–(39), the components have safety functions (43), then the system safety function is given by the vector

$$\boldsymbol{S}^1(t,\cdot) = [\boldsymbol{S}^1(t,1), \boldsymbol{S}^1(t,2),\ldots,\boldsymbol{S}^1(t,z)],$$
$$t \in <0,\infty), \quad (44)$$

with the coordinates

$$\boldsymbol{S}^1(t,u) = [S^1(t,u+1)]^n$$
$$+\int_0^t \sum_{j=1}^n [\tilde{f}^1(a,u+1)\cdot[S^1(a,u+1)]^{n-1}\cdot S^1(a,u)$$
$$\cdot\prod_{i=1}^n S^1_{i/j}(t-a,u)]da,\ \ u = 1,2,\ldots,z-1. \quad (45)$$

$$\boldsymbol{S}^1(t,z) = [S^1(t,z)]^n, \quad (46)$$

where

- $\tilde{f}^1(t,u+1)$ is the density function coordinate of a component corresponding to distribution function $\tilde{F}^1(t,u+1)$;
- $\tilde{F}^1(t,u+1)$ is the distribution function, defined as the probability of component's exit from the safety state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, before the time t, given that its lifetime in the subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$ is greater than t, is given by

$$\tilde{F}^1(t,u+1) = 1 - \frac{S^1(t,u+1)}{S^1(t,u)}; \quad (47)$$

- $S^1{}_{i/j}(t,u)$ is the safety function coordinate of the component E_i, $i = 1,\ldots,n$, after departure of the component E_j, $j = 1,\ldots,n$, from the safety state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, such that

$$S^1{}_{i/j}(t-a,u) = \frac{S^1_{i/j}(t,u)}{S^1(a,u)},\ 0 < a < t,$$
$$t \in <0,\infty),\ i = 1,\ldots,n,\ j = 1,\ldots,n. \quad (48)$$

In a particular case, when the components of the series system have piecewise exponential safety functions, defined by (43), with the coordinates of the form

$$S^1(t,u) = \exp[-\lambda^1(u)t],\ t \in <0,\infty),\ \lambda^1(u) \geq 0,$$
$$u = 1,2,\ldots,z, \tag{49}$$

where $\lambda^1(u)$, $u = 1,2,\ldots,z$, is the intensity of departure from the safety state subset $\{u, u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, of the components of the series homogeneous system before the leaving this safety state subset by any of the system components, the distribution function $\tilde{F}^2(t,u+1)$, defined by (47), is given by

$$\tilde{F}^1(t,u+1) = 1 - \exp[-[\lambda^1(u+1) - \lambda^1(u)]t],$$
$$t \in <0,\infty),\ u = 1,2,\ldots,z-1, \tag{50}$$

and its corresponding density function is

$$\tilde{f}^1(t,u+1) = \frac{d}{dt}\tilde{F}^1(t,u+1)$$
$$= [\lambda^1(u+1) - \lambda^1(u)]\exp[-[\lambda^1(u+1) - \lambda^1(u)]t],$$
$$t \in <0,\infty),\ u = 1,2,\ldots,z-1. \tag{51}$$

Further, by considering (49), the components E_i, $i = 1,2,\ldots,n$, after the departure of component E_j, $j = 1,2,\ldots,n$, from the subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, have the safety functions given by (40) with the coordinates (41)–(42) with the following forms:

$$S^1_{i/j}(\upsilon) = \exp[-\lambda^1_{i/j}(\upsilon)t],\ t \in <0,\infty),$$
$$\upsilon = u, u-1,\ldots,1,\ u = 1,2,\ldots,z-1,\ i = 1,\ldots,n,$$
$$j = 1,\ldots,n, \tag{52}$$

$$S^1_{i/j}(t,\upsilon) = \exp[-\lambda^1(\upsilon)t],\ t \in <0,\infty),$$
$$\upsilon = u+1, u+2,\ldots,z,\ u = 1,2,\ldots,z-1,$$
$$i = 1,\ldots,n,\ j = 1,\ldots,n, \tag{53}$$

where the intensities

$$\lambda^1_{i/j}(\upsilon),\ \upsilon = u, u-1,\ldots,1,\ u = 1,2,\ldots,z-1,$$
$$i = 1,\ldots,n,\ j = 1,\ldots,n,$$

are defined by (36)–(37) with

$$\lambda^1_i(u) = \lambda^1(u),\ u = 1,2,\ldots,z,\ i = 1,\ldots,n,$$

and

$$\lambda^1(u),\ u = 1,2,\ldots,z,$$

are the intensity existing in the formula (49).

Considering (48)–(53), we can obtain *Theorem 1* particular case for the homogeneous multistate ageing series system of components having piecewise exponential safety functions formulated as follows.

Corollary 1.

If, in a homogeneous multistate ageing series system with dependent components following the dependency rule (34)–(39), the components have piecewise exponential reliability functions with the coordinates (49), then the system safety function is given by the vector

$$\boldsymbol{S}^1(t,\cdot) = [\boldsymbol{S}^1(t,1), \boldsymbol{S}^1(t,2),\ldots,\boldsymbol{S}^1(t,z)],$$
$$t \in <0,\infty), \tag{54}$$

with the coordinates

$$\boldsymbol{S}^1(t,u) = \exp[-n\lambda^1(u+1)t]$$
$$+\frac{1}{n}[1 - \exp[-n[\lambda^1(u+1) - \lambda^1(u)]t]]$$
$$\cdot \sum_{j=1}^{n}\exp[-\sum_{i=1}^{n}\lambda^1_{i/j}(u)t],\ t \in <0,\infty),$$
$$u = 1,2,\ldots,z-1, \tag{55}$$

$$\boldsymbol{S}^1(t,z) = \exp[-n\lambda^1(z)t],\ t \in <0,\infty), \tag{56}$$

where

$$\lambda^1(u),\ u = 1,2,\ldots,z,$$

are the intensities of ageing of components of the homogeneous series system before the leaving the safety state subset $\{u, u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, by any of the system components, existing in the formula (49) and

$$\lambda^1_{i/j}(u),\ u = 1,2,\ldots,z-1,\ i = 1,\ldots,n,\ j = 1,\ldots,n,$$

are the intensities of ageing of components of the homogeneous series system after the leaving this safety state subset by any of the system components, defined by (35)–(39), given by (36) with

$$\lambda^1_i(u) = \lambda^1(u),\ u = 1,2,\ldots,z,\ i = 1,\ldots,n.$$

Motivation: Substituting (49), (51), (48) and (52) into (45), we get

$$\boldsymbol{S}^1(t,u)=\exp[-n\lambda^1(u+1)t]$$
$$+\int_0^t[\lambda^1(u+1)-\lambda^1(u)]\exp[-n[\lambda^1(u+1)-\lambda^1(u)]a]da$$
$$\cdot\sum_{j=1}^{n}\exp[-\sum_{i=1}^{n}\lambda^1_{i/j}(u)t],\ \ t\in<0,\infty),\ u=1,2,\dots,z-1.$$

Hence, after integration, we get (55). Next, substituting (49) for $u=z$ into (46), we get (50). This way, the motivation is completed.

4.3. Safety parameters of car wheel system with dependent components

In this section, we accept assumptions and agreements of Section 4.1 and Section 4.2 and we arbitrarily assume the following coefficients of the components dependency impact introduced by (33) and (39)

$$q(\upsilon,E_j,E_i)=d(\upsilon,d_{ij}),\ \ \upsilon=u,u-1,\dots,1,\ \ u=1,2,3,$$
$$i=1,2,3,4,\ \ j=1,2,3,4,$$

on the car wheel system components E_i, i = 1,2,3,4, lifetimes in the safety state subsets {u, u+1, …, 3}, $u=1,2,3$:

- after the exit of E_1 from the safety state subset {u, u+1, …, 3}

$$d(\upsilon,d_{11})=1,\ d(\upsilon,d_{21})=0.94,$$
$$d(\upsilon,d_{31})=0.96,$$
$$d(\upsilon,d_{41})=0.98,\ \upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_2 from the safety state subset {u, u+1, …, 3}

$$d(\upsilon,d_{12})=0.94,\ d(\upsilon,d_{22})=1,$$
$$d(\upsilon,d_{32})=0.98,$$
$$d(\upsilon,d_{42})=0.96,\ \upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_3 from the safety state subset {u, u+1, …, 3}

$$d(\upsilon,d_{13})=0.96,\ d(\upsilon,d_{23})=0.98,$$
$$d(\upsilon,d_{33})=1,$$
$$d(\upsilon,d_{43})=0.94,\ \upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_4 from the safety state subset {u, u+1, …, 3}

$$d(\upsilon,d_{14})=0.98,\ d(\upsilon,d_{24})=0.96,$$
$$d(\upsilon,d_{34})=0.94,\ d(\upsilon,d_{44})=1,$$
$$\upsilon=u,u-1,\dots,1,\ u=1,2,3. \tag{57}$$

From the above, considering (37), the coefficients of system components dependency impact on the system components E_i, i = 1,2,3,4, intensities of ageing $\lambda_i^2(u)$, before the E_j, j = 1,2,3,4, departure from the safety state subset {u,u+1,…, 3}, u = 1,2,3, are:

- after the exit of E_1 from the safety state subset {u, u+1, …, 3}

$$\rho^1_{1/1}(\upsilon)=\frac{1}{d(\upsilon,d_{11})}=\frac{1}{1},$$
$$\rho^1_{2/1}(\upsilon)=\frac{1}{d(\upsilon,d_{21})}=\frac{1}{0.94},$$
$$\rho^1_{3/1}(\upsilon)=\frac{1}{d(\upsilon,d_{31})}=\frac{1}{0.96},$$
$$\rho^1_{4/1}(\upsilon)=\frac{1}{d(\upsilon,d_{41})}=\frac{1}{0.98},$$
$$\upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_2 from the safety state subset {u, u+1, …, 3}

$$\rho^1_{1/2}(\upsilon)=\frac{1}{d(\upsilon,d_{12})}=\frac{1}{0.94},$$
$$\rho^1_{2/2}(\upsilon)=\frac{1}{d(\upsilon,d_{22})}=\frac{1}{1},$$
$$\rho^2_{3/2}(\upsilon)=\frac{1}{d(\upsilon,d_{32})}=\frac{1}{0.98},$$
$$\rho^1_{4/2}(\upsilon)=\frac{1}{d(\upsilon,d_{42})}=\frac{1}{0.96},$$
$$\upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_3 from the safety state subset {u, u+1, …, 3}

$$\rho^1_{1/3}(\upsilon)=\frac{1}{d(\upsilon,d_{13})}=\frac{1}{0.96},$$
$$\rho^1_{2/3}(\upsilon)=\frac{1}{d(\upsilon,d_{23})}=\frac{1}{0.98},$$
$$\rho^1_{3/3}(\upsilon)=\frac{1}{d(\upsilon,d_{33})}=\frac{1}{1},$$
$$\rho^1_{4/3}(\upsilon)=\frac{1}{d(\upsilon,d_{43})}=\frac{1}{0.94},$$
$$\upsilon=u,u-1,\dots,1,\ u=1,2,3;$$

- after the exit of E_4 from the safety state subset $\{u, u+1, \ldots, 3\}$

$$\rho^1_{1/4}(\upsilon) = \frac{1}{d(\upsilon, d_{14})} = \frac{1}{0.98},$$
$$\rho^1_{2/4}(\upsilon) = \frac{1}{d(\upsilon, d_{24})} = \frac{1}{0.96},$$
$$\rho^1_{3/4}(\upsilon) = \frac{1}{d(\upsilon, d_{34})} = \frac{1}{0.94},$$
$$\rho^1_{4/4}(\upsilon) = \frac{1}{d(\upsilon, d_{44})} = \frac{1}{1},$$
$$\upsilon = u, u-1, \ldots, 1,\ u = 1,2,3. \tag{58}$$

Considering (49), we assume that the car wheel system components E_i, $i = 1,2,3,4$, under their dependency, conditional safety functions (30)–(31) are piecewise exponential with the coordinates

$$S^1_i(t,u) = \exp[-\lambda^1_i(u)t],\ t \in <0, \infty),$$
$$u = 1,2,3,4,\ i = 1,2,3,4, \tag{59}$$

where the intensities of ageing

$$\lambda^1_i(u),\ u = 1,2,3,4,\ i = 1,2,3,4,$$

for the car wheel system components E_i, $i = 1,2,3,4$, under their dependency before the changing the safety state subset $\{u, u+1, \ldots, 4\}$, u = 1,2,3,4, by any of the system components are given by

$$\lambda^1_i(u) = \lambda^1(u) = \lambda^0(u),\ u = 1,2,3,4,\ i = 1,2,3,4, \tag{60}$$

where

$$\lambda^0(u),\ u = 1,2,3,4,\ i = 1,2,3,4,$$

are the intensities of ageing of the homogeneous car wheel system components E_i, $i = 1,2,3,4$, without their dependency impact and given by (18). i.e.

$$\lambda^1_i(1) = \lambda^0(1) = 0.0125,\ \lambda^1_i(2) = \lambda^0(2) = 0.0200,$$
$$\lambda^1_i(3) = \lambda^0(3) = 0.0250,\ \lambda^1_i(4) = \lambda^0(4) = 0.03125,$$
$$i = 1,2,3,4. \tag{61}$$

After that, from (36), considering (59)–(61), it follows that the intensities of the homogeneous car wheel system components departure from the safety states subset $\{u, u+1, \ldots, 3\}$, u = 1,2,3, related to their dependency impact on their safety are

$$\lambda^1_{i/j}(\upsilon) = \rho^1_{i/j}(\upsilon) \cdot \lambda^1(\upsilon) = \rho^1_{i/j}(\upsilon) \cdot \lambda^0(\upsilon),$$
$$\upsilon = u, u-1, \ldots, 1,\ u = 1,2,3,\ i = 1,2,3,4, \tag{62}$$

and particularly, according to (60)–(61) and (58), we have:

- after the exit of component E_1 from the safety state subsets {1,2,3}, {2,3}, {3}

$$\lambda^1_{1/1}(1) = \frac{1}{1} 0.0125,$$
$$\lambda^1_{1/1}(2) = \frac{1}{1} 0.0200,$$
$$\lambda^1_{1/1}(3) = \frac{1}{1} 0.0250,$$
$$\lambda^1_{2/1}(1) = \frac{1}{0.94} 0.0125,$$
$$\lambda^1_{2/1}(2) = \frac{1}{0.94} 0.0200,$$
$$\lambda^1_{2/1}(3) = \frac{1}{0.94} 0.0250,$$
$$\lambda^1_{3/1}(1) = \frac{1}{0.96} 0.0125,$$
$$\lambda^1_{3/1}(2) = \frac{1}{0.96} 0.0200,$$
$$\lambda^1_{3/1}(3) = \frac{1}{0.96} 0.0250,$$
$$\lambda^1_{4/1}(1) = \frac{1}{0.98} 0.0125,$$
$$\lambda^1_{4/1}(2) = \frac{1}{0.98} 0.0200,$$
$$\lambda^1_{4/1}(3) = \frac{1}{0.98} 0.0250;$$

- after the exit of component E_2 from the safety state subsets {1,2,3}, {2,3}, {3}

$$\lambda^1_{1/2}(1) = \frac{1}{0.94} 0.0125,$$
$$\lambda^1_{1/2}(2) = \frac{1}{0.94} 0.0200,$$
$$\lambda^1_{1/2}(3) = \frac{1}{0.94} 0.0250,$$
$$\lambda^1_{2/2}(1) = \frac{1}{1} 0.0125,$$
$$\lambda^1_{2/2}(2) = \frac{1}{1} 0.0200,$$
$$\lambda^1_{2/2}(3) = \frac{1}{1} 0.0250,$$

$$\lambda^1_{3/2}(1) = \frac{1}{0.98}0.0125,$$
$$\lambda^1_{3/2}(2) = \frac{1}{0.98}0.0200,$$
$$\lambda^1_{3/2}(3) = \frac{1}{0.98}0.0250,$$
$$\lambda^1_{4/2}(1) = \frac{1}{0.96}0.0125,$$
$$\lambda^1_{4/2}(2) = \frac{1}{0.96}0.0200,$$
$$\lambda^1_{4/2}(3) = \frac{1}{0.96}0.0250;$$

- after the exit of component E_3 from the safety state subsets {1,2,3}, {2,3}, {3}

$$\lambda^1_{1/3}(1) = \frac{1}{0.96}0.0125,$$
$$\lambda^1_{1/3}(2) = \frac{1}{0.96}0.0200,$$
$$\lambda^1_{1/3}(3) = \frac{1}{0.96}0.0250,$$
$$\lambda^1_{2/3}(1) = \frac{1}{0.98}0.0125,$$
$$\lambda^1_{2/3}(2) = \frac{1}{0.98}0.0200,$$
$$\lambda^1_{2/3}(3) = \frac{1}{0.98}0.0250,$$
$$\lambda^1_{3/3}(1) = \frac{1}{1}0.0125,$$
$$\lambda^1_{3/3}(2) = \frac{1}{1}0.0200,$$
$$\lambda^1_{3/3}(3) = \frac{1}{1}0.0250,$$
$$\lambda^1_{4/3}(1) = \frac{1}{0.94}0.0125,$$
$$\lambda^2_{4/3}(2) = \frac{1}{0.94}0.0200,$$
$$\lambda^1_{4/3}(3) = \frac{1}{0.94}0.0250;$$

- after the exit of component E_4 from the safety state subsets {1,2,3}, {2,3}, {3}

$$\lambda^1_{1/4}(1) = \frac{1}{0.98}0.0125,$$
$$\lambda^1_{1/4}(2) = \frac{1}{0.98}0.0200,$$
$$\lambda^1_{1/4}(3) = \frac{1}{0.98}0.0250,$$
$$\lambda^1_{2/4}(1) = \frac{1}{0.96}0.0125,$$
$$\lambda^1_{2/4}(2) = \frac{1}{0.96}0.0200,$$
$$\lambda^1_{2/4}(3) = \frac{1}{0.96}0.0250,$$
$$\lambda^1_{3/4}(1) = \frac{1}{0.94}0.0125,$$
$$\lambda^1_{3/4}(2) = \frac{1}{0.94}0.0200,$$
$$\lambda^1_{3/4}(3) = \frac{1}{0.94}0.250,$$
$$\lambda^1_{4/4}(1) = \frac{1}{1}0.0125,$$
$$\lambda^1_{4/4}(2) = \frac{1}{1}0.0200,$$
$$\lambda^1_{4/4}(3) = \frac{1}{1}0.0250. \quad (63)$$

4.4. Safety indicators of car wheel system with dependent components

From the results (61) and (63), applying (55)–(56), according to *Corollary 1*, the car wheel system safety function is given by

$$\boldsymbol{S}^1(t,\cdot) = [\boldsymbol{S}^1(t,1),\ \boldsymbol{S}^1(t,2),\ \boldsymbol{S}^1(t,3),\ \boldsymbol{S}^1(t,4)], \quad (64)$$

where

$$\boldsymbol{S}^1(t,1) = \exp[-4\lambda^1(2)t]$$
$$+\frac{1}{4}[1-\exp[-4[\lambda^1(2)-\lambda^1(1)]t]]\ \sum_{j=1}^{4}\exp[-\sum_{i=1}^{4}\lambda^1_{i/j}(1)t]$$
$$= \exp[-0.08t]$$
$$+[1-\exp[-0.03t]]\exp\,(1+\frac{1}{0.98}+\frac{1}{0.96}+\frac{1}{0.94})t]$$
$$\cong \exp[-0.08t]\ +[1-\exp[-0.03t]]\exp[-0.051574t],$$

$$\boldsymbol{S}^1(t,2) = \exp[-4\lambda^1(3)t]$$
$$+\frac{1}{4}[1-\exp[-4[\lambda^1(3)-\lambda^1(2)]t]]$$
$$\sum_{j=1}^{4}\exp[-\sum_{i=1}^{4}\lambda^1_{i/j}(2)t]\ = \exp[-0.10t]$$
$$+[1-\exp[-0.02t]]\exp\,(1+\frac{1}{0.98}+\frac{1}{0.96}+\frac{1}{0.94})t]$$
$$\cong \exp[-0.10t]\ +[1-\exp[-0.02t]]\exp[-0.082518t]$$

$$\boldsymbol{S}^1(t,3) = \exp[-4\lambda^1(4)t]$$
$$+\frac{1}{4}[1-\exp[-4[\lambda^1(4)-\lambda^1(3)]t]]\ \sum_{j=1}^{4}\exp[-\sum_{i=1}^{4}\lambda^1_{i/j}(3)t]$$

$$= \exp[-0.125t]$$
$$+[1-\exp[-0.025t]] \exp\left(1+\frac{1}{0.98}+\frac{1}{0.96}+\frac{1}{0.94}\right)t]$$
$$\cong \exp[-0.125t] + [1-\exp[-0.025t]]$$
$$\exp[-0.103148t],$$

$$S^1(t,4) = \exp[-4\lambda^1(4)t] = \exp[-4\cdot 0.03125t]$$
$$= \exp[-0.125t], \; t \in <0,\infty). \quad (65)$$

The graph of the car wheel system safety function is shown in *Figure 7*.

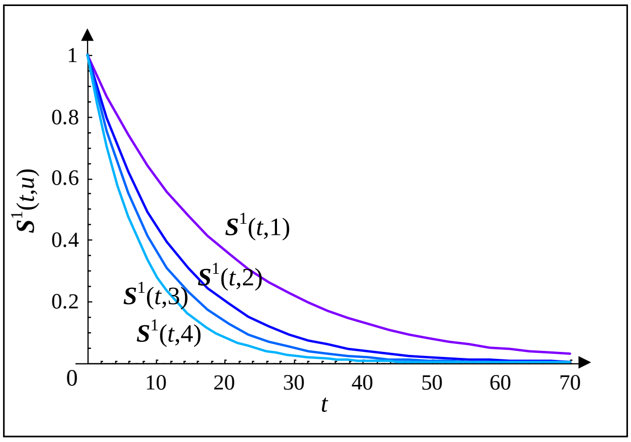

Figure 7. The graph of the car wheel system safety function $S^1(t,\cdot)$ coordinates

As the critical safety state is r = 2, then by (4) and (65), the car wheel system impacted by its components dependency risk function is given by

$$r^1(t) = 1 - S^1(t,2)$$
$$= 1 - \{\exp[-0.10t]$$
$$+[1-\exp[-0.02t]]\exp[-0.082518t]\},$$
$$t \in <0,\infty). \quad (66)$$

From the formula for the inverse risk function [8] and (66), the moment when the car wheel system impacted by its components dependency risk function exceeds a permitted level δ = 0.05 is

$$\tau^1 = r^{1^{-1}}(\delta) \cong 0.634 \text{ year}. \quad (67)$$

Considering (65) and applying suitable formulae from [8], the expected values of the car wheel system impacted by its components dependency lifetimes in the safety state subsets {1,2,3,4}, {2,3,4}, {3,4}, {4}, respectively are:

$$\mu^1(1) = \int_0^\infty S^1(t,1)dt \cong 19.631 \text{ years},$$

$$\mu^1(2) = \int_0^\infty S^1(t,2)dt \cong 12.364 \text{ years},$$
$$\mu^1(3) = \int_0^\infty S^1(t,3)dt \cong 9.891 \text{ years},$$
$$\mu^1(4) = \int_0^\infty S^1(t,4)dt = 8 \text{ years}. \quad (68)$$

Further, from the above, it follows that the mean values of the car wheel system impacted by its components dependency lifetimes in the particular safety states 1, 2, 3, 4, are:

$$\bar{\mu}^1(1) = \mu^1(1) - \mu^1(2) \cong 7.267 \text{ years},$$
$$\bar{\mu}^1(2) = \mu^1(2) - \mu^1(3) \cong 2.473 \text{ years},$$
$$\bar{\mu}^1(3) = \mu^1(3) - \mu^1(4) \cong 1.891 \text{ years},$$
$$\bar{\mu}^1(4) = \mu^1(4) = 8 \text{ years}. \quad (69)$$

The intensities of degradation (ageing)

$$\lambda^1(t,1),\; \lambda^1(t,2),\; \lambda^1(t,3),\; \lambda^1(t,4),\; t \in <0,\infty),$$

of the car wheel system impacted by its components dependency / the intensities of the car wheel system impacted by its components dependency departure from the safety state subsets {1,2,3,4}, {2,3,4}, {3,4}, {4}, can be determined according to the formula [8]

$$\lambda^1(t,u) = -\frac{dS^1(t,u)}{dt} / S^1(t,u),\; t \in <0,\infty), \quad (70)$$

where $S^1(t,u)$, $t \in <0,\infty)$, u = 1,2,3,4, are given by (65).

The graphs of the intensities of ageing of the car wheel system impacted by its components dependency, given by (70), are shown in *Figure 8*.

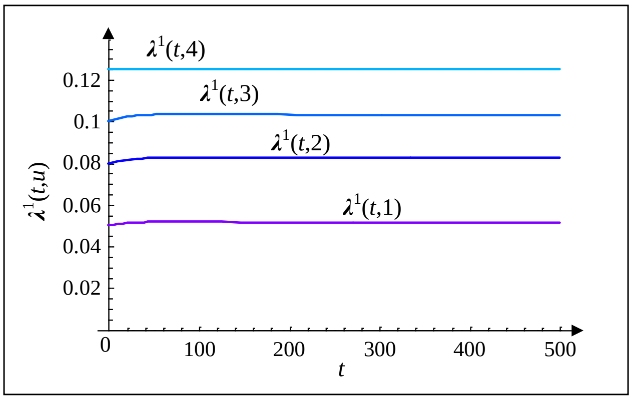

Figure 8. The graphs of the intensities of ageing of the car wheel system impacted by its components dependency

By (65), applying (70), we obtain the following limit intensities of ageing of the car wheel system impacted by its components dependency:

$$\lambda^1(1) = \lim_{t\to\infty} \lambda^1(t,1)$$
$$\cong \lim_{t\to\infty} \frac{0.051574\exp[-0.051574t][1+o(1)]}{\exp[-0.051574t][1+o(1)]} \cong 0.0516,$$

$$\lambda^1(2) = \lim_{t\to\infty} \lambda^1(t,2)$$
$$\cong \lim_{t\to\infty} \frac{0.082518\exp[-0.082518t][1+o(1)]}{\exp[-0.082518t][1+o(1)]} \cong 0.0825,$$

$$\lambda^1(3) = \lim_{t\to\infty} \lambda^1(t,3)$$
$$\cong \lim_{t\to\infty} \frac{0.103148\exp[-0.103148t][1+o(1)]}{\exp[-0.103148t][1+(1)]} \cong 0.1031,$$

$$\lambda^1(4) = \lim_{t\to\infty} \lambda^1(t,4)$$
$$= \lim_{t\to\infty} \frac{0.125\exp[-0.125t]}{\exp[-0.125t]} = 0.125. \quad (71)$$

By (68), applying suitable formulae from [8], the approximate mean intensities of ageing of the car wheel system impacted by its components dependency are:

$$\lambda^1(1) = \frac{1}{\boldsymbol{\mu}^1(1)} \cong \frac{1}{19.631} \equiv 0.0509,$$
$$\lambda^1(2) = \frac{1}{\boldsymbol{\mu}^1(2)} \cong \frac{1}{12.364} \equiv 0.0809,$$
$$\lambda^1(3) = \frac{1}{\boldsymbol{\mu}^1(3)} \cong \frac{1}{9.861} \equiv 0.1014,$$
$$\lambda^1(4) = \frac{1}{\boldsymbol{\mu}^1(4)} \cong \frac{1}{8} = 0.125. \quad (72)$$

The comparison of the results (71) and (72) with the results given by (29) proves a slight influence of the wheel system components mutual dependency on its intensities of ageing.

Considering (71) and the values of the car wheel system intensities of ageing without its components dependency impact, determined by (29), the components dependency impact on the car wheel system intensities of ageing are:

$$\boldsymbol{\rho}^1(t,1) = \frac{\lambda^1(1)}{\lambda^0(1)} \cong \frac{0.0516}{0.05} = 1.031,$$
$$\boldsymbol{\rho}^1(t,2) = \frac{\lambda^1(2)}{\lambda^0(2)} \cong \frac{0.0825}{0.08} = 1.031,$$
$$\boldsymbol{\rho}^1(t,3) = \frac{\lambda^1(3)}{\lambda^0(3)} \cong \frac{0.1031}{0.10} = 1.031,$$
$$\boldsymbol{\rho}^1(t,4) = \frac{\lambda^1(4)}{\lambda^0(4)} \cong \frac{0.125}{0.125} = 1.000. \quad (73)$$

Considering (72) and the values of the car wheel system intensities of ageing without of its components dependency impact, determined by (29), the coefficients of the components dependency impact on the car wheel system intensities of ageing are:

$$\boldsymbol{\rho}^1(t,1) = \frac{\lambda^1(1)}{\lambda^0(1)} \cong \frac{0.0509}{0.05} \cong 1.018$$
$$\boldsymbol{\rho}^1(t,2) = \frac{\lambda^1(2)}{\lambda^0(2)} \cong \frac{0.0809}{0.08} \cong 1.018,$$
$$\boldsymbol{\rho}^1(t,3) = \frac{\lambda^1(3)}{\lambda^0(3)} \cong \frac{0.1014}{0.10} = 1.014,$$
$$\boldsymbol{\rho}^1(t,4) = \frac{\lambda^1(4)}{\lambda^0(4)} \cong \frac{0.125}{0.125} = 1.000. \quad (74)$$

Finally, by (73) and (74), the car wheel system resilience indicator, i.e. the coefficient of the car wheel system resilience to its components dependency impact, is given either by

$$\boldsymbol{RI}^1(t,2) = \frac{1}{\boldsymbol{\rho}^1(t,2)} \cong \frac{1}{1.031} \cong 0.97 = 97\%, \quad (75)$$

or by

$$\boldsymbol{RI}^1(t,2) = \frac{1}{\boldsymbol{\rho}^2(t,2)} \cong \frac{1}{1.011} \cong 0.99 = 99\% \quad (76)$$

The comparison of safety indicators (20)–(29) and (64)–(72) proves a slight influence of the inside components dependency on the car wheel system safety what is also clearly expressed in the resilience indicators to its components dependency impact (73)–(76). This slight influence follows from the arbitrarily assumed small values of the coefficients of impact of components dependence on their intensities of ageing.

5. Conclusion

In the chapter, the approach to the safety analysis of multistate ageing systems that considers their components' dependency is presented. In the safety analysis of multistate ageing systems, it is assumed that the degradation of safety state of one or a group of components may cause a degradation of the condition and safety parameters of other components, and consequently affect the functioning and lifetime of the entire system. From a practical point of view this assumption is very important as, in real practice, often not only the system lifetime depends on its possible shortening caused by changes in the safety states of its components, but also its lifetime depends on the dependences among its components ageing and changing their safety state subsets. Combining the results of the safety analysis of a multistate ageing system with its components dependency, the safety analysis of a series multistate ageing system considering its inside dependences is performed and the new results that improve significantly the accuracy of the real system safety examination are found.

The generalization of the obtained new results to safety analysis of multistate ageing networks with cascading effects [1] at their operation conditions [5] is a very broad topic with a multidimensional problem and many issues still need to be analysed and resolved. The analysis of interdependencies among infrastructures at their operation conditions and dependencies among assets inside infrastructures is very complicated due to the problem of determining in real complex networks where and how these relationships occur, what is their type is, whether they depend on other external factors and a number of other practically important questions. Answering these questions and building one general approach covering them jointly seems to be very challenging [6].

Referring to cascading effect in infrastructure networks at their operation conditions, the issue of initials that cause degradation and further cascade damage in the networks, is also raised. The importance of the place in the network structure in which the initial damage appeared and the strength of this destruction should be emphasized. And consequently, very important for critical infrastructure operating environment safety and security, the problems of modelling, identification, prediction and mitigation [3] of the critical infrastructure accident consequences are arising.

Another important issue regarding dependency analysis in critical networks at their operation conditions is how information about interactions among components within the network and among subnetworks is used for critical infrastructure networks design and operation management and make them more resistant to disturbances, degradation and failures of other components and subnetworks.

Thus, as a consequence of the above analysis, the further initial steps in research could be focused on safety analysis of complex multistate ageing systems [8] and critical infrastructure networks [11], considering jointly [6] their ageing [8], inside dependencies [1] and external impacts [5], and the use of the achieved results to improve their safety [8], strengthen their resilience and mitigate [3] the effects of their degradation and failures.

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