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Reliability based models to support risk management decision making

Keywords

reliability, risk, assessment, decision making

Abstract

The chapter has been intended to provide a comprehensive understanding on one class of reliability models used to perform probabilistic risk assessments. This is the class of binary-based conceptual models. This class of models allows assessing events occurrence, system states identification, states transitions and corresponding relevant probabilistic dynamic quantities. Besides, it allows the binarization of multistate systems so that ever bigger systems can be modelled, under conditions. The term "system" is used in its widest sense to include physical engineering systems, processes, and any structured set of actions/events. Consequently, the modelling capabilities cover sequential events and cycling transitions. The move from "risk management decision-making" to "risk-informed decision-making" paradigm has obviously created the right environment to develop and implement risk-based decision-making models in a variety of sectors. A brief and short non-exhaustive survey is presented identifying sectors using these models to assess risks in support to decision-making.

1. Introduction

Reliability Models (RMs) provide major support to decision-making processes in a large variety of sectors. They provide decision makers with rationales that are traceable, transmissible, and formal. They are subject to continual improvement and maturation thanks to the cumulated operational feedback experiences. The last interesting point of RMs and not the least, they can be encoded in numerical programs and processed by computers. Subsequently, big-data processing, automation, self-learning, and artificial-intelligence algorithms become daily realities in almost all sectors of Man's activities.

Acknowledging all what mentioned above, the major attractiveness of the RMs for decision-making results from their dynamic-modelling capability and subsequently their predictive potential [18]–[19], [20], [28], [41].

This is precisely what a decision-maker is seeking for [17], [21], [27], [35]. In this work we focus on one set of RMs. Decision making issue is briefly treated in Section 2 and Section 3. The specificity of decision making in relation with risk management is treated in Section 2. While the evolution of the paradigms of risk management is treated in Section 3. Three paradigms are identified in Section 3: deterministic, coupled deterministic-uncertainty, and the probabilistic ones. The probabilistic paradigm used in risk-based decision making is exploiting the full predictive and dynamic potential of the RMs.

In this work we focus on one macro-class of RMs, the class of logical-numerical models. Within this macro-class, we focus on one specific class which is the binary-based reliability models.

Within the frame of the binary-based models, Section 4, one starts with a basic elementary binary (*EB*) system which is the simplest conceptual functional entity one may conceive (Section 4.1) the analysis of the *EB* system reveals a mathematical pattern that governs the system dynamics (Section 4.2). This mathematical pattern consists of three differential equations describing the time behavior of the three quantities characterizing the *EB* system: the probability to be in a given state at time '*t*', the probability of sojourning in that state within an interval 'Δ', and the probability of sojourning out of that state within an interval 'Δ'. Then we use the *EB* system to build up the multistate system (Section 5) and so that we may move closer to real life engineering systems. Then, we present, briefly, two tools to represent the multistate systems: the Boolean tool (Section 5.1) and the state graph one

Safety and Reliability of Systems and Processes, Summer Safety and Reliability Seminar 2020

DOI: 10.26408/srsp-2020-06

(Section 5.2). While the interstate transitions and the independency condition are treated in Section 5.3. Once after, the multistate system is binarized back again (Section 6) to serve in building a system of systems, and so on. We demonstrate that the multistate system is also characterized by the same quantities as the *EB* system (Section 6.1, Section 6.2, Section 6.3). The cycle from an elementary binary system, to a multistate one and back to a (non-) elementary binary system again is governed by the same generic mathematical dynamic pattern (Section 6.4). This pattern is described by a set of three differential equations describing respectively the three probabilistic quantities that fully describe the system dynamic: the probability to be in a well-defined state or set of stats of interest, the sojourn probability in that state or set of stats, and the sojourn probability out of that state or set of stats.

We would like to underline that the term system covers all forms of organized (/structured) actions (/activities) that are designed to realize a well-defined objective or set of objectives.

Besides, the terms 'state' and 'set of states' can be described by Boolean cut-sets (minimal or not) and by a graph of states.

Once the system is defined and the states of interest are identified, the transitions between the states will be determined. The dynamic aspect of the RMs comes from the full description of these transitions either through differential equations or integral ones.

The structurally related subjects, that are widely modelled with the binary based RMs, are: the transitions cycling and the sequential events. The first subject is treated in Section 7 while the later is treated in Section 8. Dynamic sequential modelling is treated in Section 8.1 and static modelling is treated in Section 8.2. The static model is considered as a special case of the dynamic on.

A short survey of the sectorial exploitations of the RMs is given is Section 9. The short survey is far from being considered as exhaustive. However, the author believes is representative enough of the sectorial end-users of the RMs.

To size in which manner these models can support decision-making, we should, first, identify the issue of "decision-making".

2. What is the issue in decision making?

What is the real issue in "decision-making"?

Is it the "decision" by itself or the "subject of the decision"?

Waking up in the morning and thinking of what decision(s) to be taken today, one may think of thousands of decisions need be taken. But no one ever thought of taking a decision such as:

1. Should I step back to "Yesterday"?,
2. Should I step forward to "Tomorrow"?, or
3. Should I stay in "Today"?

This decision has simply non-sense to be worked out. Decision making has a sense when there is more than one option. Even with more than one option, decision-makers can still make the best decision if and only if all the options, the profit function of each option and the constraints in each option are well-determined and static. One will naturally choose the most beneficial option or the less harmful one. Operational Research and optimization algorithms were just other formal approaches to treat that kind of decisions.

Nevertheless, decision-making becomes a real concern if: "the options and the corresponding conditions are dynamic and, additionally, random".

Indeed, this is the case with real life systems and processes created by man.

Facing this issue of decision making, risk assessment and decision-making has evolved in three phases over almost 2 centuries. Three phases and three risk relevant decision-making paradigms.

3. Reliability model and decision making

Three major paradigms have sequentially been introduced on the scene of risk assessment and risk-informed decision making:

- deterministic,
- coupled deterministic-uncertainty,
- probabilistic (objective and subjective).

These three paradigms have chronologically been developed in the order given above.

3.1. Deterministic paradigm

By the 2nd half of the 19th century, the biggest technical risky challenges were related to civil and mechanical engineering realisation. The success or the failure of an engineering object was expressed by a trade-off between the resistance of the used materials and the strength of the applied loads. We may call that the Stress-Resistance (*S*–*R*) paradigm. Although, the original inspiration of the model is from material sciences, it could easily serve in a variety of engineering fields as varying as: finances, psychology, sociology, ecology, history of civilisations, …, etc.

The *S*–*R* paradigm in its simplest formulation: "if the stress is higher than the resistance, the system fails".

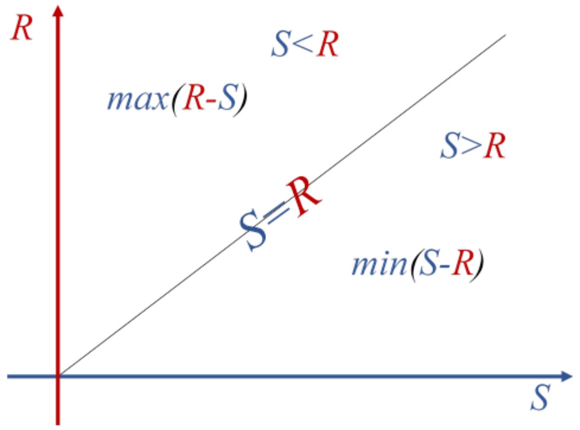

Figure 1. Schematic representation of the *S–R* risk paradigm

One can then identify three operational regimes:

- if $S < R$ a safe regime,
- if $S = R$ a critical regime,
- if $S > R$ a failure regime.

So, risk manager and engineers had one ultimate objective is to maximize (*R–S*) if $R > S$ or to minimize (*S–R*) if $S > R$.
Once that is stated, engineering objects and processes were designed so that the operational loads are one third of the critical load so that the used materials may support the load. The paradigm gave birth to the famous "Safety Factor".
And the Safety Factor based risk management became the mainstream.

3.2. Coupled deterministic uncertainty

While the civil and mechanical realisations continued their ever-increasing development, applied statistics also continue its progress and tools development.
The continual increase of successful civil and mechanical engineering realizations was naturally accompanied with an increasing number of failure cases, as well. Despite of the use of the safety factor, some realizations failed. Something was still uncertain but happening and leading to the collapse of some nice and big engineering achievements, during the first half of the 20th century.
In the meantime, applied statistics was expanding its assessments in epidemiology, insurance, banking, and stock market decision making activities [4], [11], [21]. A statistics-based paradigm was developing with obvious growing concerns related to risk assessment, risk management and risk prediction.
The specific statistical concept of "uncertainty" started invading hard fundamental and engineering sciences, especially systems engineering.
The fully deterministic paradigm was moving to a coupled deterministic-uncertainty one. Risk assessments continue to be performed based on deterministic models. The uncertainties were injected into the input data.
Engineers, designers, and analysts admitted that the values of the materials resistance and of the applied stresses are mean values and are associated with a given distribution that has its own inherent dispersion characteristics (variance and other different moments). By commodity, in most of the engineering applications the used distribution function is a Gaussian one, *Figure 2*.
The paradigm moved from deterministic (*S–R*) to a deterministic coupled with uncertainty assessment.

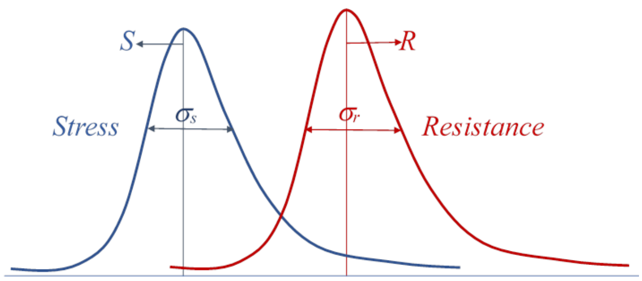

Figure 2. Schematic representation of the coupled uncertainties *S–R* risk paradigm

Practices are moving step by step from risk-management to risk-management and uncertainty-informed decision-making.

3.3. Probabilistic paradigm

Starting from the second half of the 20th century. Rapid progress in sciences and technology gave birth to illimited types of realizations with more complex systems, smarter and interactive. Besides, the operational feedback from previous engineering realizations was also cumulating at high rate producing excessively big operational databases. The statistical treatment of these data allowed to identify patterns and constructing probabilistic models of failure modes and initiating events of different kinds.
In many engineering disciplines, deterministic thinking admitted and increasingly integrated statistical-probabilistic thinking [2], [24].
Probabilistic assessments are diffusing almost in all daily life activities [17]–[18], [28]. Even in the most orthodox deterministic engineering disciplines such as structure mechanics and rupture mechanics.
Risk ideological thinking is now increasing and shifting to risk-informed decision-making.
Probabilistic Risk Assessment (PRA) informs decision-maker.
PRA finds its foundation in the Reliability Theory and exploits Reliability Models (RMs). Then, lets briefly present a selected choice of some of these RMs, exclusively, those that are binary based

models. We start first by the most basic elementary conceptual model that may ever exist.

4. Elementary binary system

The Elementary Binary (EB) is a conceptual binary functional entity, *Figure 3*.

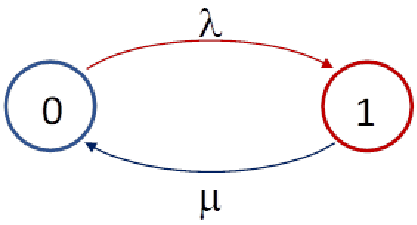

Figure 3. A schematic representation of the *EBS*

The *EBS* has only two possible and exclusive states. The system spends his lifetime in successive transitions from one state to the other. The transitions from one stat to the other are described by a well-defined random process, characterized by two transition rates $(\lambda(t), \mu(t))$. The transition rates can generally be any function of time.
Let the state of interest be the state (0), one can then determine at any instant of time the probability to be in state (0), as following:
"Let $p(t + \Delta t)$ be the probability to be in state (0) at $(t + \Delta t)$. The EBS can be in state (0) if and only it was already at state (0) at "t" and had not gone through any transition out of the state within the interval "Δt", or it was not in the state (0) at "t" and had gone through a transition to the state (0) within the interval "Δt". That can shortly be expressed by the following mathematical expression:

$$p(t + \Delta t) = p(t)(1 - \lambda(t)\Delta t) + \big(1 - p(t)\big)\mu(t)\Delta t, \qquad (1)$$

where:

- $p(t)(1 - \lambda\Delta t)$ is the probability to be at state (0) at time "t" and does not go through a transition out of the state (0) within the interval "Δt";
- $(1 - p(t))\mu\Delta t$ is the probability of not being at state (0) at time "t" and goes through a transition to the state (0) within the interval (Δt);
- $p(t + \Delta t)$ is the probability to be at state (0) at time "$t + \Delta t$".

Having taken the limit $(\Delta t \to 0)$, (1) becomes:

$$\frac{d}{dt}p(t) = -\big(\lambda(t) + \mu(t)\big)p(t) + \mu(t). \qquad (2)$$

One can certainly use the same procedure regarding the probability $q(t)$ of "not being in state (0)" at time "t". We recall that both $p(t)$ and $q(t)$ are expressing two instantaneous qualities, i.e, both are determined at instant "t". Both verify the following relation at any instant "t":

$$p(t) + q(t) = 1, \quad \forall t \in \mathcal{R}^+.$$

The solution of (2) requires an initial condition. Analysts used to use the initial condition $q(0)$ rather than $p(0)$ and assign the letter "γ" for this initial condition, $\gamma = q(0) = 1 - p(0)$. In the case of mechanical systems reliability, "γ" is called "failure-to-start" probability. We insist on the fact that "γ" is a probability while $(\lambda(t), \mu(t))$ are transition rates.
Two other quantities are used to specify the dynamic of the systems: the "sojourn probability in state (0)" and the "sojourn probability out of state (0)". Both are integral qualities, i.e., are determined over a time-interval "$T; T = t - t_1$" and $t \geq t_1$.
Let's $R(t + \Delta t, t_1)$ be the sojourn probability in state (0) continuously within the period $(T;\ T = t + \Delta t - t_1)$. As "$t_1$" is a fixed lower limit of the period, one can write the sojourn probability as $R(t + \Delta t)$, for the sack of simple writings.
The procedure to determine the sojourn probabilities is basically similar to the one described above for the instantaneous probability.
Let "$R(t + \Delta t)$" be the sojourn probability in state (0) continuously over the whole interval $(t + \Delta t)$. Then, one can write:

$$R(t + \Delta t) = R(t) \cdot (1 - \lambda\Delta t),$$

where

- $R(t)$ is the probability that the system has already sojourned continuously during the whole interval "$t - t_1$";
- $(1 - \lambda\Delta t)$ is the probability to continue for an additional infinitesimal period Δt.

Having developed the left-hand side of the equation and taken the limit $\Delta t \to 0$, one finds that:

$$\frac{d}{dt}R(t) = -\lambda R(t). \qquad (3)$$

The same can exactly be done to determine the sojourn probability out of state (0), $S(T)$. That gives:

$$\frac{d}{dt}S(t) = -\mu S(t). \qquad (4)$$

It is worth emphasizing on the differences between the probabilities of being in/out of a given state $p(t)/\,q(t)$ and the probability of sojourning in a given state $R(T(t))/S(T(t))$. The probabilities of being in/out of a given state are instantaneous quantities at instant "t". While, the probabilities of sojourning in/out of a given state are integral quantities determined over an interval "t".

4.1. Solution pattern

The general solutions of equations (2) for the instantaneous probability $p(t)$, the sojourn-in probability $R(t)$ and the sojourn-out probability $S(t)$ have respectively the following form:

$$p(t)\cdot e^{\int_0^t \sigma(\xi)d\xi} = \left(\int_0^t \mu(\xi)\cdot e^{\int_0^{\xi}\sigma(\eta)d\eta}d\xi\right) + p(0), \quad (5)$$

where

$$\sigma(t) = \lambda(t) + \mu(t)$$

and

$$p(0) = 1 - \gamma.$$

While the solutions for $R(t)$ and $S(t)$ are given by:

$$R(t) = e^{-\int_0^t \lambda(\xi)d\xi},$$

$$S(t) = e^{-\int_0^t \mu(\xi)d\xi}.$$

Given that the characteristic parameters are well-defined, $(\lambda(t), \mu(t), \gamma)$.

Equations (2) may sometimes have analytical solutions. That is driven by the nature of the stochastic processes that govern the transitions from and to the states of interest. If the transitions are governed by a stochastic Poisson process, i.e., the transition rates are time-independent, then the analytical solutions exist for all the concerned quantities: $p(t)$, $q(t)$, $R(t)$ and $S(t)$.

4.2. Asymptotic behaviour

From equations (2), one can obviously deduce that this dynamic predictive model shows an asymptotic behavior such as:

$$\lim_{t\to\infty} p(t) = \frac{\mu(t\to\infty)}{(\lambda(t\to\infty) + \mu(t\to\infty))}.$$

The asymptotic limit of $p(t)$ exists if and only if one of the characteristic parameters (λ, μ) is not null.

The same for the sojourn probabilities:

$$\lim_{t\to\infty} R(t) \to 0.$$

The asymptotic value of $\lim_{t\to\infty} R(t)$ vanishes if and only if $\lambda(t\to\infty)$ does not vanish, and similarly:

$$\lim_{t\to\infty} S(t) \to 0.$$

The asymptotic value of $\lim_{t\to\infty} S(t)$ vanishes if and only if $\mu(t\to\infty)$ does not vanish.

4.3. Dynamic and stochastic pattern

Equations (2) describe perfectly the dynamic random process governing any elementary binary-based system concept.

Obviously, this conceptual pattern is dynamic, predictive, and able to handle random processes.

The issue now is: *can this conceptual model be used to construct more elaborated models describing systems closer to real-life ones, given that real systems are very often multi state ones?*

5. Multistate system

To keep in line with the didactic characteristic of the paper, lets immediately choose a simple academic case to apply on.

Let MS be a simple multistate system that includes three independent elementary binary systems (EBS). The success of this MS requires at least the success of 2 out of the three EBS: $s1$, $s2$, and $s3$. The overall system success state S can then be determined by the following Boolean expression:

$$S = s1\cdot s2 + s2\cdot s3 + s3\cdot s1, \quad (6)$$

where '$\cdot$' and '+' are the Boolean operators 'AND' and 'OR', respectively. The success is logically described in (6) using the minimal cut-sets of "success".

Besides, this multi-state system can also be represented by its corresponding graph of states, as well, *Figure 4*.

The Boolean representation in (6) and the graphical one in *Figure 4* are equivalent.

The success space of MS contains 4 states (in green background) that verify the success criteria given by the Boolean expression in (6). The other 4 states (in red background) do not.

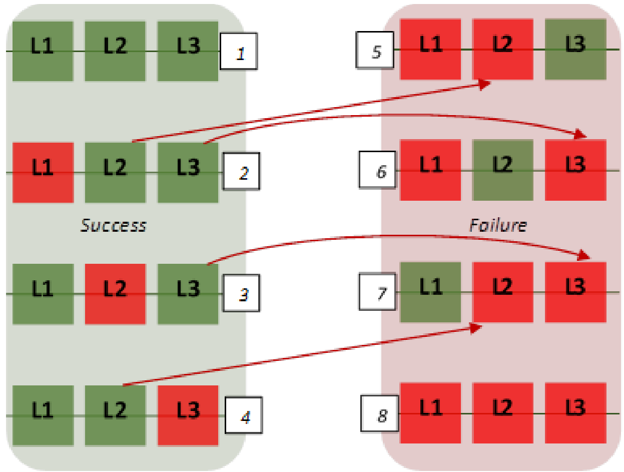

Figure 4. State graph of a 2/3 multi-state system (success = green, non-success = red)

It may be necessary to underline that the equal sharing of the states between the success and the failure spaces is specific to the 2/3 case. Otherwise, the sharing of the states is quite different from one multistate system to another. It is obvious that functional states sharing is driven by the success / non-success imposed criteria.
The probability of the success of the *MS* can then be determined using the two models, the Boolean, and the state-graph one.

5.1. Boolean based model

The Boolean expression of success (6) can be rewritten using its exclusive cut-sets rather than the minimal cut-sets. As there are three minimal cut-sets in (6), then we may expect having 6, (3! = 6), alternative but equivalent expressions with exclusive cut-sets. One of them looks like the following expression:

$$S = s1 \cdot s2 + \overline{s1} \cdot s2 \cdot s3 + \overline{s2} \cdot s3 \cdot s1, \qquad (7)$$

where $s1$ describes the success of system 1 and $\overline{s1}$ describes the non-success (the failure).
As the expression in (7) contains only exclusive cut-sets, so the overall success probability is directly the sum of the occurrence probabilities of each of the exclusive cut-sets. Accordingly, the probability of success $P_S(t)$ can immediately be determined by:

$$P_S(t) = p_1(t) \cdot p_2(t) + q_1(t) \cdot p_2(t) \cdot p_3(t) + q_2(t) \cdot p_3(t) \cdot p_1(t). \qquad (8)$$

While $Q_S(t)$ is given by:

$$Q_S(t) = q_1(t) \cdot q_2(t) + p_1(t) \cdot q_2(t) \cdot q_3(t) + p_2(t) \cdot q_3(t) \cdot q_1(t), \qquad (9)$$

where;

$$p_i(t) + q_i(t) = 1, \quad i = 1, 2, 3,$$

$p_i(t)$ is the solution of (2) for each of the three independent elementary success, using the corresponding parameters $(\lambda_i(t), \mu_i(t), \gamma_i)$ of each.
It may certainly be worth underling that both Boolean and state graph models are helpful to directly determine the instantaneous dynamic quantities like $p_i(t)$ and $q_i(t)$. However, it is not as straight forward regarding the determination of the integral quantities like $R_i(T(t))$ and $S_i(T(t))$. We will come to this point later: (11) (12).

5.2. Graph based model

One can also use the state-graph model to determine the probability of success as long as all the corresponding parameters $(\lambda_i(t), \mu_i(t), \gamma_i)$ are known.
If the corresponding parameters $(\lambda_i(t), \mu_i(t), \gamma_i)$. are time-constant, the state-graph will be a Markov type graph describing a stochastic Poisson process. If the corresponding parameters $(\lambda_i(t), \mu_i(t), \gamma_i)$ are slowly varying with time, one can use semi-Markov linearization approach [21]–[22].
Otherwise, one may use Monte Carlo simulation.
However, the probabilities of success and failure are not the only characteristic quantities of a multistate system. Analysts can determine many other characteristic quantities, such as: the transition rates between states or set of states, and the transition rates between the success and the failure spaces.
But before proceeding to the transitions between different states, we should briefly cover a subject relating the impact of transitions' independency on the transitions.

5.3. Transitions and independency

A founding assumption in this binary-based set of models is the "transitions independency" or the "independent occurrence" of the transitions.
Take any two different independent transitions on the state graph, characterized by their transition rates "$\tau_1(t)$" and "$\tau_2(t)$", respectively. The individual probabilities of these transitions within a time-interval "Δt" are respectively "$\tau_1(t)\Delta t$" and "$\tau_2(t)\Delta t$". Given that the two transitions are independent, then the probability of observing both transitions in the same interval of time is simply $(\tau_1(t)\Delta t \cdot \tau_2(t)\Delta t)$. Besides, let "$T(t)$" be the rate of the simultaneous occurrence of both transitions at time "t", then the simultaneous occurrence

probability within the time-interval "Δt" can be determined by "$T(t)\Delta t$", as well.
One can then lay down the necessary condition:

$$T(t)\Delta t = \tau_1(\mathrm{t})\Delta t \cdot \tau_2(\mathrm{t})\Delta t \ .$$

Clearing Δt off from both sides and taking the limits $\Delta t \to 0$:

$$T(t) = \lim_{\Delta t \to 0}[(\tau_1(\mathrm{t}) \cdot \tau_2(\mathrm{t}))\Delta t] \ .$$

Having taken the limits, we get the following:

$$T(t)\big)_{\Delta t \to 0} \ \to \ 0.$$

That simply means that the rate of a simultaneous occurrence of independent events is always null.
The rule is that "independent events can't happen simultaneously". Simultaneous independent transitions are forbidden.
Only dependent/correlated transitions can happen simultaneously.
This is a characteristic quality in all multistate systems built up with independent transitions.
Once the forbidden transitions notion is established, we may proceed to the multistate system binarization.

6. Multistate system binarization

One can also determine some other characteristic qualities of multistate systems. That requires acquiring the notions of "critical transitions", "critical states" and the "multistate system binarization".
One can see in *Figure 4* that the transitions between the space of success and the space of failure go through some paths. These paths design the "critical transitions". Critical transitions are, then, those that result in systematic transitions from one space to another. Not all the interstate transitions are then critical.
On the state-graph in *Figure 4*, one can determine 12 possible transitions in the success space between the states 1, 2, 3 and 4, in both senses. None of them is a critical transition. One may notice the same for the space of failure.
However, one can determine only 12 critical transitions in both senses between these 6 couples of states: (2,5), (2,6), (3,5), (3,7), (4,6) and (4,7). Notice that the states (2,3,4) belong to the success space while states (5,6,7) belong to the failure space. That is why these states and their corresponding transitions are critical.
On *Figure 4*, One should have a total of 56 potential transitions in both senses between all the states:

$$2 \cdot \binom{8}{2} = 2 \cdot 28 = 56,$$

then, we still have 20 non-active transitions in both senses, $(56 - 3 \cdot 12 = 20)$.
These transitions are forbidden because of the hypotheses of the "independence" between transitions laid down above. This is the case, for example, of the transitions between the couple of states (1,7).
Once critical transitions are identified in both senses between the success and the failure spaces, one may envisage the determination of the "equivalent transition rates" in both senses between the spaces of success and failure.
The determination of such equivalent transition rates for the whole system ($\Lambda(t)$, $M(t)$) is straight forward once; the concerned states and their transitions are determined.
Considering the system represented by the state graph given in *Figure 4*, one can determine the "equivalent transition rate out of the success space", $\Lambda(t)$. To proceed towards this target, one need determining the probability $P_S(t)$ to be in the success state, first.
The system probability $P_S(t)$ to be in the success space at time "t" is determined directly from the state graph, such as:

$$\begin{aligned} P_S(t) \ &= \ p_1(t) \cdot p_2(t) \cdot p_3(t) \\ &+ q_1(t) \cdot p_2(t) \cdot p_3(t) \\ &+ p_1(t) \cdot q_2(t) \cdot p_3(t) \\ &+ p_1(t) \cdot p_2(t) \cdot q_3(t). \end{aligned} \quad (10)$$

Notice that (10) which is directly deduced from the state graph is equal to (9) deduced directly from the logical expression of success, despite the slightly different form of each.
Having determined the $P_S(t)$, we can proceed to the determination of the system equivalent parameters ($\Lambda(t)$, $M(t)$, Γ).

6.1. Equivalent initial condition

Having determined the probability of the system to be in the success space, $P_S(t)$, at any instant of time "t", the probability of not to be in the success space at time "$t = 0$", $Q_S(0)$, can immediately be determined from (9) as:

$$\begin{aligned} \Gamma_S &= \gamma_1 \cdot \gamma_2 + (1 - \gamma_1) \cdot \gamma_2 \cdot \gamma_3(t) \\ &+(1 - \gamma_2) \cdot \gamma_3 \cdot \gamma_1 \end{aligned}$$

using (10) will give the same result.

6.2. Equivalent transition rates

Once $P_s(t)$ is determined by one way or the other, one can proceed to the determination of $\Lambda(t)$ as following. Let $\Lambda(t)$ be the equivalent transition rate of the whole system from the success space to the failure one.
The occurrence probability of the transition within an interval "Δt" is given by $(\Lambda_s(t)\Delta t \cdot P_s(t))$, where $P_s(t)$ is the probability to be in the success space at "t". This transition probability is also equal to the sum of all the probabilities describing the critical transitions from the success space, such as:

$$\begin{aligned}&\Lambda_s(t)\Delta t \cdot P_s(t)\\&= (\lambda_2 + \lambda_3)\Delta t \cdot q_1(t) \cdot p_2(t) \cdot p_3(t)\\&+ (\lambda_1 + \lambda_3)\Delta t \cdot p_1(t) \cdot q_2(t) \cdot p_3(t)\\&+ (\lambda_1 + \lambda_2)\Delta t \cdot p_1(t) \cdot p_2(t) \cdot q_3(t).\end{aligned}$$

Clearing "Δt" from both sides and taking the limits, we get:

$$\begin{aligned}&\Lambda_s(t) \cdot P_s(t)\\&= (\lambda_2 + \lambda_3) \cdot q_1(t) \cdot p_2(t) \cdot p_3(t)\\&+ (\lambda_1 + \lambda_3) \cdot p_1(t) \cdot q_2(t) \cdot p_3(t)\\&+ (\lambda_1 + \lambda_2) \cdot p_1(t) \cdot p_2(t) \cdot q_3(t).\end{aligned} \tag{11}$$

Determining the system equivalent transition rate back, $M(t)$, to the success space is straight forward as above. It is determined as following:

$$\begin{aligned}&M_s(t) \cdot Q_s(t)\\&= (\mu_2 + \mu_3) \cdot p_1(t) \cdot q_2(t) \cdot q_3(t)\\&+ (\mu_1 + \mu_3) \cdot q_1(t) \cdot p_2(t) \cdot q_3(t)\\&+ (\mu_1 + \mu_2) \cdot q_1(t) \cdot q_2(t) \cdot p_3(t).\end{aligned} \tag{12}$$

By that stage, the multistate system is fully characterized by its parameters $(\Lambda_s(t), M_s(t), \Gamma_s)$ at any instant of time.

6.3. Sojourn probabilities

Once the equivalent parameters are determined, one can proceed to the determination of the sojourn-in and the sojourn-out probabilities $(R_s(t), S_s(t))$, such as:

$$R_s(t) = \exp\left(-\int_0^t \Lambda_s(\xi)d\xi\right)$$

and

$$S_s(t) = \exp\left(-\int_0^t M_s(\xi)d\xi\right).$$

The determination of the above integrations requires very often the use of numerical techniques. However, by no means this can be considered as a serious issue.

6.4. Inclusion in the pattern

The system equivalent parameters $(\Lambda(t), M(t), \Gamma)$ are the characteristic parameters of the whole system. Subsequently, they obey to the same pattern given in equations (2) such as:

$$\frac{d}{dt}P_s(t) = -(\Lambda(t) + M(t))P_s(t) + M(t), \tag{13}$$

$$\frac{d}{dt}R_s(t) = -\Lambda(t)R_s(t), \tag{14}$$

$$\frac{d}{dt}S_s(t) = -M(t)S_s(t), \tag{15}$$

given the initial condition "Γ_S".
Once the multistate system is binarized, it can be then handled as if it was an elementary binary system and combined with other elementary binary systems to build up bigger and bigger systems.
Caution: The ever building up bigger and bigger systems is valid if and only if all the used elementary binary systems (basic or not) are independent.

6.5. Asymptotic behaviour

From the pattern given in (13)-(15), one can deduce that these systems have an asymptotic behavior when "t" is big enough $(t \to \infty)$. At the asymptotic the characteristic quantities of the system become as such:

$$P_s(t)]_{t\to\infty} = \left(\frac{\mathrm{M}(t)}{\mathrm{M}(t)+\Lambda(t)}\right)_{t\to\infty} \tag{16}$$

and

$$Q_s(t)]_{t\to\infty} = \left(\frac{\Lambda(t)}{\mathrm{M}(t)+\Lambda(t)}\right)_{t\to\infty}. \tag{17}$$

The asymptote of $P_s(t)$ ($/Q_s(t)$) exists if and only if, at least, one of the asymptotic values of the transition rates does not tend to zero.
While the asymptote values of the sojourn probabilities tend to zero:

$$R_s(t)]_{t\to\infty} \to 0,$$

$$S_s(t)]_{t\to\infty} \to 0.$$

If and only if the asymptotic value of the corresponding transition rate does not tend to zero.
By now, we have shown by so far that the multistate

system is fully described by its characteristic parameters $(\Lambda(t), M(t), \Gamma)$ in the same pattern as the elementary binary one with its ones $(\lambda(t), \mu(t), \gamma)$. Consequently, we will use now and then the triplet $(\lambda(t), \mu(t), \gamma)$ to characterize both the elementary binary systems and the multistate ones.

7. Transition cycles

In many sectorial applications, analysts would like to analyze and assess the occurrence of some well-defined transition-cycles within a given interval of time.

A transition cycle is defined as a transition out of and then back to a given set of states. Let $\mathcal{S}$ be a set of states of interest (success), fully defined by its characteristic transition rates out and in: λ and μ, respectively. Both λ and μ are generally function of time.

The probability $P_k(t)$ that the system experiences k transition cycles within the interval $[0, t]$ is given by the following integral equation [14]:

$$P_k(t) = \int_{\xi=0}^{t} \int_{\eta=\xi}^{t} [\,P_{k-1}(\xi)](\lambda d\xi) e^{-\mu(\eta-\xi)} (\mu d\eta) e^{-\lambda(t-\eta)}. \qquad (18)$$

If the transition rates λ and μ are constant with time, (18) analytical solution exist:

$$P_k(t) = \Psi_k(t) e^{-\lambda t} - \Phi_k(t) e^{-\mu t}, \qquad (19)$$

where

$$\Psi_k(t) = \left(\frac{\lambda\mu}{\sigma^2}\right)^k \left[\sum_{j=0}^{k} (-1)^j \cdot C_j^k \cdot \frac{(\sigma t)^{k-j}}{(k-j)!}\right], \qquad (20)$$

$$\Phi_k(t) = (-1)^k \cdot \left(\frac{\lambda\mu}{\sigma^2}\right)^k \left[\sum_{j=0}^{k} B_j^k \cdot \frac{(\sigma t)^{k-j}}{(k-j)!}\right] \qquad (21)$$

and

$$\sigma = (\mu - \lambda),$$

$$C_0^k = 1\,, \qquad B_0^k = 0, \qquad k \geq 0,$$

$$C_k^k = B_k^k\,, \qquad B_k^k = C_{k-1}^k + B_{k-1}^k, \qquad k \geq 1,$$

$$C_{j-1}^k = C_{j-2}^k + C_{j-1}^{k-1}, B_{j-1}^k = B_{j-2}^k + B_{j-1}^{k-1}, k \geq 1.$$

In *Figure 5*, the time profile of $P_k(t)$ is determined with the given transition rates value mentioned on the figure caption for different values of k.

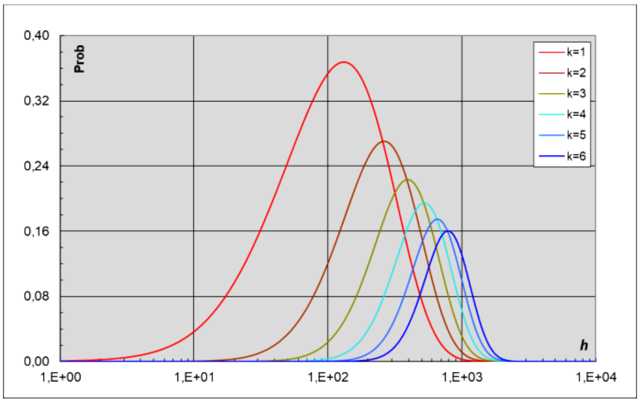

Figure 5. $P_k(t)$ time profile for transition rates $(\lambda = 8^{-3}/h, \mu = 1.5^{-1}/h)$

The assessment of the number of transitions out of and into a given set of states, within a given interval of time, is of great interest in many applications. The model opens the possibility to determine, within a given interval of time, other probabilistic quantities such as: the expected number of cycles, the most probable number of cycles, the least probable number of cycles and the meantime between successive cycles.

8. Sequential transitions

It is more than frequent that big crises occur as results of undesired and failure sequential events.

Analysts are then interested in assessing sequential events. We may distinguish two different approaches to perform sequential event analyses: a dynamic approach and a static one.

Both approaches may schematically be represented on the form of event trees. Both can also be represented mathematically by integrations.

8.1. Dynamic sequential assessment

The occurrence probability of a sequence of transitions occur in that order $\{E_1, E_2, E_3, \ldots, E_n\}$ and their occurrence times are $\{t_1, t_2, t_3, \ldots, t_n\}$, such as $\{t_1 < t_2 < t_3 < \ldots < t_n < t\}$ can be described by the following integral equations [12]:

$$P_n(t) = \int_0^t \rho_1(t_1) dt_1 \int_{t_1}^t \rho_2(t_2) dt_2 \ldots \int_{t_{n-1}}^t \rho_n(t_n) dt_n, \qquad (22)$$

where $\rho_i(t)$ is the density probability function of the transition i, $i = 1,2, \ldots, n$.

This model is dynamic as the occurrence times may happen at any instant of time between 0 and t.

The only restriction is to occur in the given sequential order $\{t_1 < t_2 < t_3 < \dots < t_n < t\}$.
If the occurrence rates of all the transitions are constant in time, i.e., all the transition are obeying to Poisson Stochastic Processes characterized by their time-constant transitions rates $\{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$, the solution of (12) have the following form:

$$P_n(t) = \sum_{j=1}^{n} C_j^n \cdot \left(1 - e^{-\left(\sum_{l=n-j+1}^{n} \lambda_l\right)t}\right), \quad (23)$$

where

$$C_1^1 = 1,$$

$$C_1^{i+1} = \sum_{j=1}^{i} C_j^i,$$

$$C_{j+1}^{i+1} = -\frac{\lambda_i}{\sum_{l=i-j+1}^{i+1} \lambda_l} C_j^i,$$

for all $j = 1, 2, \dots, i$ and $i = 1, 2, \dots, n$.

In *Figure 6*, the time profile of the occurrence probability of a given sequence composed of four transitions characterized by their occurrence rates $\{8 \cdot 10^{-1},\ 4 \cdot 10^{-1},\ 2 \cdot 10^{-1},\ 1 \cdot 10^{-1}/h\}$.
Results are traced for two occurrence reversal orders: the occurrence rates decreasing order (blue-line) and the increasing order (red-line).

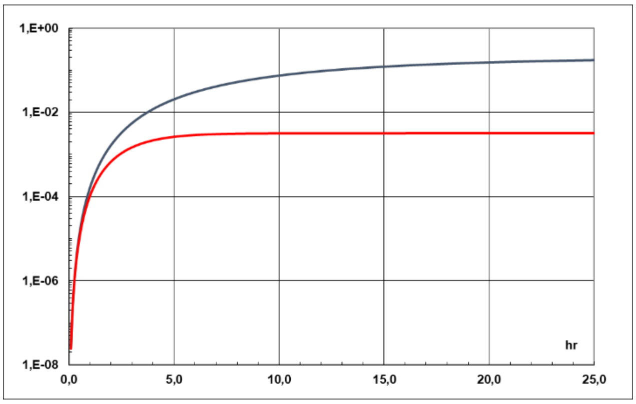

Figure 6. $P_n(t)$ of the sequence $\{E_1, E_2, E_3, E_4\}$ in two reversal orders

8.2. Static sequential model

In the case of the static sequential modelling, the sequential transitions still occur one after the other. But each transition happens within a prefixed time-interval Δ.
In this case, the sequence $\{E_1, E_2, E_3, \dots, E_n\}$ has fixed occurrence time-intervals $\{\Delta_1, \Delta_2, \Delta_3, \dots, \Delta_n\}$.
Given the density probability functions of the transitions, $\rho_i(t)$ $(i,\ i = 1,2, \dots, n)$, the occurrence probability $P_n(t)$ of the sequence $\{E_1, E_2, E_3, \dots, E_n\}$ will be described by the following integral equations:

$$P_n(t) = \int_0^{\Delta_1} \rho_1(t_1)dt_1 \int_{\Delta_1}^{\Delta_1+\Delta_2} \rho_2(t_2)dt_2 \dots \int_{\sum_1^{n-1}\Delta_i}^{\sum_1^n \Delta_i} \rho_n(t_n)dt_n. \quad (24)$$

Having fixed the integral interval of each transition separately, the (24) can then be written as:

$$P_n(t) = \left(\int_0^{\Delta_1} \rho_1(t_1)dt_1\right)\left(\int_{\Delta_1}^{\Delta_1+\Delta_2} \rho_2(t_2)dt_2\right) \dots \left(\int_{\sum_1^{n-1}\Delta_i}^{\sum_1^n \Delta_i} \rho_n(t_n)dt_n\right).$$

Finally, the integral takes the following form:

$$P_n(t) = \prod_{i=1}^{n} p_i(\Delta_i). \quad (25)$$

The sequential model is reduced into the form of an independent occurrence of each event 'i' within its own interval 'Δ_i'.

9. Sectorial end-users

To demonstrate the sectorial extension of the reliability-based models to support decision-making, we give four heterogenous sectorial applications in the following.
In neuroscience the "optimal decision-making" problem constitutes making optimal use of sensory information that mind processes. One typical example of decision-making within a dynamic and random conditions is driving a car. The reliability models are used to approach the patterns of mind reasoning [5], [8]–[10].
In AI and automation field for clinical applications, probabilistic decision making is also exploiting and developing reliability-based models in view of developing smart diagnosis algorithms [1], [3], [6]–[7], [25], [40].
In drinking water network design and maintenance, RMs are used to enhance the water quality, to detect and predict aging and to optimize maintenance and periodic testing [32]–[34].
For Critical Infrastructure Protection (CIP), the RMs are developed to model and assess the CIP resilience [13], [23], [31].
RMs are also used in design decision-making, especially, if the design object is the first-of-the kind. The assessment of a future fusion power reactor is one case [15].

Table 1. Paradigms of the most used models to support decision-making per sector

#	Sector	Deterministic	Determ.+Uncertain.	Probabilistic
1	Accident Analysis		x	x
2	Aeronautic + Spatial Design		x	x
3	Behavioural Analysis & Profiling		x	x
4	Blockchain Management			x
5	Civil Engineering Conventional Design	x	x	
6	Civil Engineering Offshore Design	x	x	x
7	Crisis Management		x	x
8	Critical Infrastructure Protection	x	x	x
9	Customs Inspection Activities		x	x
10	Epidemiological Crisis Management		x	x
11	Financial Risk & Stock Market Management		x	x
12	Hydraulic Works & Circuits Des.	x	x	
13	Information Analytics & Intelligence		x	x
14	Insurance		x	x
15	Maintenance Optimisation	x	x	x
16	Periodic Testing		x	x
17	Pharmaceutical & Drugs Testing		x	x
18	Preventive Maintenance		x	x
19	Quality Control		x	x
20	Random Mechanical Loading on Structures		x	x
	Total	**5**	**19**	**18**

In the nuclear power plants safety and accident consequences prediction, the mythical WASH-740 [39], studied the possible consequences of a hypothetical major accident on Long Island New York reactor (1957) but the used models were inadequate. Then WASH-1400 on "Reactor Safety Study" was published on 1975 for updating purpose by introducing a reliability-based models [38]. WASH-1400 was then updated by NUREG-1150 [30]. Last updating is the report NRC-ML1734 (2019) [29]. NRC-ML1734 used deterministic-integrated uncertainty and probabilistic models.

In the aviation sector, the AFF developed probability-based regulations to enhance safety [16]. Besides, EU research & innovation programs finance activities to develop probabilistic models to support risk-based decision making in view of enhancing the aviation sector operational safety [36].

Today, probabilistic risk-informed decision-making in normal operational and anormal operational situations are the most used in nuclear, aeronautics and security sectors.

Table 1 presents a non-exhaustive sample of sectors that use reliability models to perform assessments supporting their decision-making activities. The sectors are classified according to the paradigms of the used models not by the used models.

The three paradigms used in the classification are: the deterministic, deterministic uncertainty-informed, and probabilistic.

It is very instructive to note that only 5 sectors over 20 use deterministic models (not exclusively) for risk assessments & management.

It is equally notable that only one sector uses exclusively the (full) probabilistic approach. This the Blockchain management sector.

We underline lastly that the most used paradigm is the coupled deterministic-uncertainty one but not exclusively with a score of 19/20, *Table 1*.

10. Conclusion

This paper is intended to give a comprehensive note on reliability-based models, exclusively, the binary conceptual models. The implementation of the reliability-based models in a risk-based decision-making process enhances its dynamic and predictive capabilities.

However, with the accelerating growing of Man's engineering knowledge and technologies, systems become more and more complex in all its dimensions: conceptual, functional, smartness, distributed, and compliance with societal concerns about risk & safety. The management of this growing complexity requires developing RMs that can compose and decompose the system at different level of its complexity without losing its dynamic characteristics. These dynamic characteristics are stored in the triplet $(\lambda(t), \mu(t), \gamma)$ that characterize every system whatever its complexity level is. This is the reason of focusing exclusively on the binary concept models to manage the growing system complexity. This binary vision is straight forward once a "success function" or a "utility function" is fixed for the multistate real systems.

However, the author is fully aware of the possibility that a multistate system may have more than one "success function" within a given mission and with the same interval of time. The answer to this situation is also straight forward using the binary based RMs, even if we have not treated this point in the paper for the sake of treating the most direct issues with the binary RMs. These direct issues are those related and defined by the minimal probabilistic and dynamic characteristic necessary to satisfy the differential equation pattern given precedingly.

The paper focuses on the potential of the binary conceptual models, besides its probabilistic and dynamic characteristics, to assess the following:

- the probability to be in/out of a given state of a set of states of interest at time "*t*",
- the sojourn probability in/out of a given state of a set of states of interest within a time interval "*T*",
- the occurrence probability of a well-defined sequence of transitions within a time interval "*T*",
- the probability the nth occurrence of a well-defined transitions-cycling within a time interval "*T*".

The paper was purposely limited to theses four dynamic and predictive quantities. Their sectorial applicative potentiality is almost unlimited. Some of these sectors have already been mentioned above.

The potential applicability of the RMs is not indeed the only factor to explain their sectorial expansion. But this expansion is also explained by the growing technical and technological complexity of the modern systems.

Developing advanced RMs is the right response to the pressing demand induced by the growing complexity of modern systems.

Acknowledgement

This work is supported by Eid Consultant – RiskLyse with the collaboration and the support of SSARS and ESReDA.

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